13 Matrices

The following notes came from Foundation mathematics (MATH 123).

Although matrices are not part of what would normally be considered "foundation mathematics", they are one of the first topics you will need in first year science and economics units. This topic is a quick introduction to matrices emphasising matrix multiplication and inverses.

A **matrix** is a rectangular array of numbers arranged in rows and columns. It is usual to enclose the array in brackets. An example of a matrix is

$$P = \begin{bmatrix} 1.918 & 1.831 \\ 3.654 & 3.528 \\ 1.470 & 1.457 \\ 0.642 & 0.889 \end{bmatrix}.$$

We often need to refer to the rows or columns of a matrix.

$$\begin{array}{cl} \begin{bmatrix} 1.918 & 1.831 \\ 3.654 & 3.528 \\ 1.470 & 1.457 \\ 0.642 & 0.889 \end{bmatrix} & \begin{array}{l} \longleftarrow \text{ row 1} \\ \longleftarrow \text{ row 2} \\ \longleftarrow \text{ row 3} \\ \longleftarrow \text{ row 4} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{col. 1} & \text{col. 2} \end{array} & \end{array}$$

We say that a matrix is an $m \times n$ matrix, or has size $m \times n$ if it has m rows and n columns. For example, the matrix $\mathbf{P}$ above is a 4×2 matrix, the matrix $\mathbf{A}$ below is a 3×4 matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -5 & 7 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & -9 & -2 \end{bmatrix}.$$

A specific element of a matrix is named by specifying its row and column. Thus the $(2, 3)$ **element** or **component** of a matrix is the number in the 2nd row and 3rd column, in the matrix $\mathbf{A}$ above the $(2, 3)$ element is 2. The components of a matrix are distinguished by subscripts — because a matrix is a two dimensional array we need two subscripts to specify a component of a matrix. Thus we refer to the $(2, 3)$ element of a matrix $\mathbf{A}$ as $A_{2\,3}$. In the matrix $\mathbf{A}$ above we have, for example:

$$A_{1\,2} = 3 \quad \text{and} \quad A_{3\,4} = -2.$$

Remember that the first index refers to the row, the second index to the column.

With this notation a general $m \times n$ matrix $\mathbf{A}$ has the form:

$$\mathbf{A} = \begin{bmatrix} A_{1\,1} & A_{1\,2} & \ldots & A_{1\,n} \\ A_{2\,1} & A_{2\,2} & \ldots & A_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m\,1} & A_{m\,2} & \ldots & A_{m\,n} \end{bmatrix}.$$

A $n \times 1$ matrix, that is a matrix with n rows and one column, is called a **column vector**. For example

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

A $1 \times n$ matrix, that is a matrix with 1 row and n columns, is called a **row vector**. For example

$$\mathbf{X} = \begin{bmatrix} X_1 & X_2 & \ldots & X_n \end{bmatrix}$$

Note that column and row vectors need only one index to label their components.

13.1 Addition and subtraction of matrices

Addition and subtraction of matrices is done by adding or subtracting corresponding components. It can only be done when the matrices are the *same size*. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 6 \\ 3 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & -3 & -1 \\ 1 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}.$$

Then

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1+0 & 0-3 & 3-1 \\ 2+1 & -1+2 & 6+3 \\ 3-1 & 1-1 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 3 & 1 & 9 \\ 2 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1-0 & 0-(-3) & 3-(-1) \\ 2-1 & -1-2 & 6-3 \\ 3-(-1) & 1-(-1) & 0-0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 1 & -3 & 3 \\ 4 & 2 & 0 \end{bmatrix}.$$

Example

Suppose numbers of three species of kangaroo are counted in three different areas over a period. Initially the numbers are represented in a matrix $\mathbf{P}$:

$$\begin{array}{cc}
 & \text{species} \\
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\mathbf{P} = & \begin{bmatrix} 460 & 54 & 210 \\ 830 & 42 & 365 \\ 670 & 63 & 288 \end{bmatrix}
\end{array}
\begin{array}{l} 1 \\ 2 \quad \text{area} \\ 3 \end{array}$$

A matrix $\mathbf{B}$ (which gives the number of births during the period), and a matrix $\mathbf{D}$ (which gives the number of deaths during the period) are given by:

$$\mathbf{B} = \begin{bmatrix} 42 & 6 & 30 \\ 88 & 7 & 43 \\ 72 & 6 & 22 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 24 & 3 & 22 \\ 103 & 12 & 83 \\ 32 & 4 & 16 \end{bmatrix}.$$

To find the change in population for each species in each area, it is natural to subtract the entries in $\mathbf{D}$ from the corresponding entries in $\mathbf{B}$ (since population change = births − deaths). We obtain

$$\mathbf{B} - \mathbf{D} = \begin{bmatrix} 18 & 3 & 8 \\ -15 & -5 & -40 \\ 40 & 2 & 6 \end{bmatrix}.$$

(This tells us, for instance, that in area 1 the 3rd kangaroo species increased by 8. The fact that the numbers in the 2nd row are all negative means that in area 2 all species of kangaroo suffered a decline in number.) To find the total numbers of kangaroos of each species in each area at the end of the period, it is natural to add the entries of $\mathbf{P}$ to the corresponding entries of $\mathbf{B} - \mathbf{D}$. We obtain

$$\mathbf{P} + (\mathbf{B} - \mathbf{D}) = \begin{bmatrix} 478 & 57 & 218 \\ 815 & 37 & 325 \\ 710 & 65 & 294 \end{bmatrix}.$$

Thus, for example, there are 710 of species 1 in area 3 at the end of the experimental period.

13.2 Multiplication of a matrix by a number

Each component of the matrix is multiplied by the number. If **A** is the matrix above, then

$$2 \times \mathbf{A} = 2 \times \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 6 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 \times 1 & 2 \times 0 & 2 \times 3 \\ 2 \times 2 & 2 \times -1 & 2 \times 6 \\ 2 \times 3 & 2 \times 1 & 2 \times 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ 4 & -2 & 12 \\ 6 & 2 & 0 \end{bmatrix}.$$

Multiplication of a matrix by a number is called **scalar multiplication**.

13.3 Sigma Notation

Sigma notation is used as a shorthand way of writing sums. If a vector **X** (either a row or column vector) has N components, then

$$\sum X_i \qquad \text{stands for} \qquad X_1 + X_2 + X_3 + \cdots + X_N.$$

Alternative notations

$$\sum X_i = \sum_i X_i = \sum_{i=1}^{N} X_i$$

are used when we want to be explicit about the index over which the summation is performed or the number of elements of the vector.

If **X** is the column vector

$$\mathbf{X} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

then

$$\sum X_i = 1 + 2 + 3 + 4 + 5 = 15$$

.

13.4 Products of Vectors

We will look at multiplication of general matrices in the next section. Here we examine the simple case of the product of a row vector and a column vector.

Given a row vector **X** and a column vector **Y** each with the same number of components, their product is

$$\mathbf{X} \times \mathbf{Y} = \sum X_i Y_i,$$

or

$$\mathbf{X} \times \mathbf{Y} = X_1 Y_1 + X_2 Y_2 + \cdots + X_n Y_n.$$

If

$$\mathbf{X} = \begin{bmatrix} 1 & 3 & 7 \end{bmatrix} \qquad \text{and} \qquad \mathbf{Y} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$$

then

$$\begin{aligned} \mathbf{X} \times \mathbf{Y} &= X_1 Y_1 + X_2 Y_2 + X_3 Y_3 \\ &= (X_1 \times Y_1) + (X_2 \times Y_2) + (X_3 \times Y_3) \\ &= (1 \times 3) + (3 \times -4) + (7 \times 2) \\ &= 3 - 12 + 14 \\ &= 5 \end{aligned}$$

Note that this definition only makes sense if the two vectors have the same number of components.

13.5 Matrix Multiplication

A steelworks uses the raw materials hematite, dolomite and coking coal to produce cast iron, iron and steel. The amounts of raw materials required for each product are given in following table.

	cast iron	iron	steel
hematite	3	3	3
dolomite	1	1	1.5
coking coal	10	14	19

All the quantities above are measured in tonnes; for example, to produce one tonne of iron requires 3 tonnes of hematite, 1 tonne of dolomite and 14 tonnes of coking coal. The data in the table can be arranged as a 3×3 matrix:

$$\mathbf{Q} = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 1.5 \\ 10 & 14 & 19 \end{bmatrix}$$

Now suppose the steelworks produces the following quantities of steel products.

	July	August
cast iron	900	1100
iron	1200	1500
steel	650	750

This data can be arranged as a 3×2 matrix

$$\mathbf{P} = \begin{bmatrix} 900 & 1100 \\ 1200 & 1500 \\ 650 & 750 \end{bmatrix}$$

We want to know what quantities of raw materials to order each month; that is, we want to fill in the following table.

	July	August
hematite	?	?
dolomite	?	?
coking coal	?	?

The unknown numbers form a 3×2 matrix $\mathbf{R}$ whose components can be worked out like this:

$$\begin{aligned} \text{hematite required in July} &= \text{hematite required to produce one tonne of cast iron} \\ &\quad\ \times \text{tonnes of cast iron produced in July} \\ &\quad + \text{hematite required to produce one tonne of iron} \\ &\quad\ \times \text{tonnes of iron produced in July} \\ &\quad + \text{hematite required to produce one tonne of steel} \\ &\quad\ \times \text{tonnes of steel produced in July} \\ &= 3 \times 900 + 3 \times 1200 + 3 \times 650 \\ &= 8250. \end{aligned}$$

Similar calculations for the other components of $\mathbf{R}$ give

$$\begin{aligned}
\mathbf{R} &= \begin{bmatrix} 3\times 900+3\times 1200+3\times 650 & 3\times 1100+3\times 1500+3\times 750 \\ 1\times 900+1\times 1200+1.5\times 650 & 1\times 1100+1\times 1500+1.5\times 750 \\ 10\times 900+14\times 1200+19\times 650 & 10\times 1100+14\times 1500+19\times 750 \end{bmatrix} \\
&= \begin{bmatrix} 2700+3600+1950 & 3300+4500+2250 \\ 900+1200+975 & 1100+1500+1125 \\ 9000+16800+12350 & 11000+21000+14250 \end{bmatrix} \\
&= \begin{bmatrix} 8250 & 10050 \\ 3075 & 3725 \\ 38150 & 46250 \end{bmatrix}.
\end{aligned}$$

So, for example, 3075 tonnes of dolomite are required in July and 46250 tonnes of coking coal in August.

You can see from this calculation that the components of $\mathbf{R}$ are obtained from the components of $\mathbf{Q}$ and $\mathbf{P}$ as follows:

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \\ R_{31} & R_{32} \end{bmatrix} = \begin{bmatrix} Q_{11}P_{11}+Q_{12}P_{21}+Q_{13}P_{31} & Q_{11}P_{12}+Q_{12}P_{22}+Q_{13}P_{32} \\ Q_{21}P_{11}+Q_{22}P_{21}+Q_{23}P_{31} & Q_{21}P_{12}+Q_{22}P_{22}+Q_{23}P_{32} \\ Q_{31}P_{11}+Q_{32}P_{21}+Q_{33}P_{31} & Q_{31}P_{12}+Q_{32}P_{22}+Q_{33}P_{32} \end{bmatrix}.$$

We write

$$\mathbf{R} = \mathbf{QP},$$

the **product** of the matrices $\mathbf{Q}$ and $\mathbf{P}$.

The value of R_{31} can be written in sigma notation like this:

$$R_{31} = \sum_k Q_{3k}P_{k1}.$$

In general

$$\begin{aligned}
R_{ij} &= \sum_k Q_{ik}P_{kj} \\
&= Q_{i1}P_{1j} + Q_{i2}P_{2j} + Q_{i3}P_{3j},
\end{aligned}$$

where i can be 1, 2 or 3 and j can be 1 or 2.

The product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined like this in the general case: Suppose $\mathbf{A}$ is an $m\times n$ matrix and $\mathbf{B}$ is an $n\times p$ matrix. Then $\mathbf{A}$ has m *rows* each of length

n and $\mathbf{B}$ has p columns each of length n. Since the rows of $\mathbf{A}$ have the same dimension as the columns of $\mathbf{B}$ it is possible to form the product of any row of $\mathbf{A}$ with any column of $\mathbf{B}$. Now the **product** $\mathbf{AB}$ is the $m \times p$ matrix $\mathbf{C}$ whose (i, j) component is the product of ith row of $\mathbf{A}$ with jth column of $\mathbf{B}$, thus

$$C_{ij} = \sum_k A_{ik} B_{kj}.$$

Another way to look at the same thing is as follows: to find the element C_{ij} run across the ith row of $\mathbf{A}$ and down the jth column of B multiplying and adding as you go

$$\mathbf{C} \qquad = \qquad \mathbf{A} \qquad \times \qquad \mathbf{B}$$

$$\begin{array}{cc} & \begin{array}{c} \mathrm{j}th \\ \text{column} \\ \downarrow \end{array} \\ \mathrm{i}th\text{ row } \rightarrow & \begin{bmatrix} & \vdots & \\ \cdots & \times & \cdots \\ & \vdots & \end{bmatrix} \end{array} = \mathrm{i}th\text{ row} \begin{bmatrix} \rightarrow & \rightarrow & \rightarrow \end{bmatrix} \times \begin{array}{c} \mathrm{j}th \\ \text{column} \\ \\ \begin{bmatrix} \quad \downarrow \quad \\ \downarrow \\ \downarrow \end{bmatrix} \end{array}$$

You perform this operation systematically, starting with the top lefthand element C_{11} which is computed from the first row of $\mathbf{A}$ and the first column of $\mathbf{B}$. Then compute C_{12} which is obtained from the first row of $\mathbf{A}$ and the second column of $\mathbf{B}$. Continue computing the first row of the product $\mathbf{C}$ by running across the first row of $\mathbf{A}$ and down successive columns of $\mathbf{B}$. Once the first row of the product is completed, start on the second row. The first element of this row C_{21} is computed from the second row of $\mathbf{A}$ and the first column of $\mathbf{B}$. The rest of the second row of the product is found by sticking with the second row of $\mathbf{A}$ and running down successive columns of $\mathbf{B}$. Continue in this way until the matrix product is complete. Although this procedure may seem complicated, it becomes fairly easy with enough practice.

When can matrices be multiplied?

To multiply $\mathbf{A}$ and $\mathbf{B}$ we form the products of the rows of $\mathbf{A}$ with columns of $\mathbf{B}$. If $\mathbf{A}$ has n columns then its rows have n components:

$$\text{no. of components of a row of } \mathbf{A} = \text{number of columns of } \mathbf{A}.$$

Similarly

$$\text{no. of components of a column of } \mathbf{B} = \text{number of rows of } \mathbf{B}.$$

Since we can form the product of two vectors only if they have the same dimension

$$\boxed{\begin{array}{c}\text{the product } \mathbf{AB} \text{ can be formed only if} \\ \text{number of columns of } \mathbf{A} = \text{number of rows of } \mathbf{B}.\end{array}}$$

The fact that not all pairs of matrices can be multiplied fits in with similar facts we already know:

- Two vectors can be added or subtracted only if they have the same dimension.
- The product of two vectors can be formed only if they have the same number of components.
- Two matrices can be added or subtracted only if they are the same size.

The order of matrix multiplication can't be changed

Suppose that the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$, so that the product $\mathbf{AB}$ can be formed. Usually $\mathbf{BA} \neq \mathbf{AB}$, so it is a mistake to change the order of multiplication in a matrix calculation. Here are some of the things that can happen.

1. **BA** doesn't exist.

   Example

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}.$$

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \times 0 + 0 \times 2 & 1 \times 1 + 0 \times 3 & 1 \times (-1) + 0 \times 0 \\ 1 \times 0 + (-1) \times 2 & 1 \times 1 + (-1) \times 3 & 1 \times (-1) + (-1) \times 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 + 0 & 1 + 0 & (-1) + 0 \\ 0 + (-2) & 1 + (-3) & (-1) + 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & -1 \end{bmatrix}.
\end{aligned}$$

However the product **BA** is not defined since **B** has 3 columns but **A** has 2 rows, and these numbers are not the same.

2. **BA** exists but is not the same size as **AB**.

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 0 & -1 \end{bmatrix}.$$

Then

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 2 \\ -2 & -3 \end{bmatrix} \end{aligned}$$

while

$$\begin{aligned} \mathbf{BA} &= \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 & 0 \\ -2 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix}. \end{aligned}$$

We see from this that **AB** and **BA** are not the same size (so can't be equal) unless **A** and **B** are "square" matrices of the same size.

3. **AB** and **BA** are the same size, but not equal.

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then

$$
\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1\times 0+1\times 1 & 1\times 1+1\times 2 \\ 2\times 0+0\times 2 & 2\times 1+0\times 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}
\end{aligned}
$$

but

$$
\begin{aligned}
\mathbf{BA} &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0\times 1+1\times 2 & 0\times 1+1\times 0 \\ 1\times 1+2\times 2 & 1\times 1+2\times 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 5 & 1 \end{bmatrix}.
\end{aligned}
$$

Q. Why make matrix multiplication so hard? Why not just multiply component-by-component as for adding and subtracting matrices?

A. If we defined matrix multiplication any other way it might be easier to work out but it wouldn't give the right answers to questions like our steelworks problem.

Multiplication of a matrix by a column vector

For example if

$$
\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}
$$

we can calculate $\mathbf{A} \times \mathbf{X}$, since the number of columns of $\mathbf{A}$ is 4 which is the same as the number of rows of $\mathbf{X}$. The result is a column vector with 3 components.

$$
\begin{aligned}
\mathbf{A} \times \mathbf{X} &= \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times 0 + 1 \times 1 + 2 \times 2 + 3 \times 3 \\ 2 \times 0 + 3 \times 1 + 4 \times 2 + 5 \times 3 \\ 3 \times 0 + 4 \times 1 + 5 \times 2 + 6 \times 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 + 1 + 4 + 9 \\ 0 + 3 + 8 + 15 \\ 0 + 4 + 10 + 18 \end{bmatrix} \\
&= \begin{bmatrix} 14 \\ 26 \\ 32 \end{bmatrix}.
\end{aligned}
$$

In general if $\mathbf{A}$ is a $m \times n$ matrix and $\mathbf{X}$ is a n component column vector, then the column vector $\mathbf{Y} = \mathbf{A} \times \mathbf{X}$ has m components given by

$$Y_i = \sum_j A_{ij} \times X_j.$$

If $\mathbf{Y} = \mathbf{A} \times \mathbf{X}$ in the example above, for instance,

$$
\begin{aligned}
Y_2 &= A_{21} \times X_1 + A_{22} \times X_2 + A_{23} \times X_3 + A_{24} \times x_4 \\
&= 2 \times 0 + 3 \times 1 + 4 \times 2 + 5 \times 3 \\
&= 26.
\end{aligned}
$$

13.6 The Identity Matrix

The $n \times n$ matrix with ones along the diagonal and zeroes everywhere else is called the $n \times n$ **identity matrix** and denoted by $\mathbf{I}$:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

If $\mathbf{I}$ is the $n \times n$ identity matrix then

$$\mathbf{I} \times \mathbf{A} = \mathbf{A}$$

for every $n \times m$ matrix $\mathbf{A}$. To see why this is so first notice that multiplying an $n \times n$ matrix by an $n \times m$ matrix gives an $n \times m$ matrix, so $\mathbf{I} \times \mathbf{A}$ is an $n \times m$ matrix, the same size as $\mathbf{A}$. Now look at the (i, j) component of $\mathbf{I} \times \mathbf{A}$, which is the product of the ith row of $\mathbf{I}$ with the jth column of $\mathbf{A}$. The ith row of $\mathbf{I}$ is is zero except for the ith component which is 1, so the vectors in the inner product are

$$\begin{bmatrix} 0 & 0 & \dots & 0 & \underbrace{\overset{\displaystyle 1}{\uparrow}}_{i\text{th component}} & 0 & \dots & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{i-1\,j} \\ A_{ij} \\ A_{i+1\,j} \\ \vdots \\ 0 \end{bmatrix}.$$

The only non-zero term in the inner product is $1 \times A_{ij}$ and therefore the (i, j) component of $\mathbf{I} \times \mathbf{A}$ is A_{ij}, which is also the (i, j) component of $\mathbf{A}$. Doing this for every i and j shows that every component of $\mathbf{I} \times \mathbf{A}$ is the same as the corresponding component of $\mathbf{A}$. So $\mathbf{I} \times \mathbf{A} = \mathbf{A}$.

You can work out similarly that if $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{B}$ is an $m \times n$ matrix then

$$\mathbf{B} \times \mathbf{I} = \mathbf{B}.$$

So *multiplication of any matrix by an identity matrix leaves the original matrix unchanged* (provided the matrices are the right sizes for the multiplication to be done).

A special case is when $\mathbf{X}$ is an n component columnvector and $\mathbf{I}$ is the $n \times n$ identity matrix, then

$$\mathbf{I} \times \mathbf{X} = \mathbf{X}.$$

Another important special case is when $\mathbf{A}$ is a square $n \times n$ matrix and $\mathbf{I}$ is the $n \times n$ identity matrix. In this case the products $\mathbf{I} \times \mathbf{A}$ and $\mathbf{A} \times \mathbf{I}$ are both defined and

$$\mathbf{I} \times \mathbf{A} = \mathbf{A} \times \mathbf{I} = \mathbf{A}.$$

This is one of the few cases where changing the order does not change the product of the matrices.

13.7 Inverse Matrices

We now know how to add, subtract and multiply matrices. It would be nice to be able to divide them too. To divide numbers it is enough to be able to find $1/a$ for every number a, because

$$b \div a = b \times (1/a).$$

For example, $5 \div 2 = 5 \times 1/2 = 2.5$. The number $1/a$ can also be written as a^{-1} and is called the **reciprocal** or **inverse** of a. It satisfies

$$a^{-1} \times a = 1.$$

The $n \times n$ matrix that behaves like the number 1 is the identity matrix $\mathbf{I}$. So to divide $n \times n$ matrices we need, for every $n \times n$ matrix $\mathbf{A}$, an $n \times n$ **inverse matrix** $\mathbf{A}^{-1}$ with

$$\mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}.$$

Because changing the order of matrix multiplication may make a difference we should also ask for

$$\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$$

as a separate request.

Actually, not every number has a reciprocal — 0 doesn't because the division $1/0$ can't be done. With numbers 0 is the only exception, but *there are many square matrices that don't have an inverse* not just matrices with all their components 0. However, most square matrices do have an inverse, so it is well worth finding out about inverses.

The inverse of a 2×2 matrix

Only for 2×2 matrices is it easy to tell whether there is an inverse and, if so, find it. Inverses of square matrices of any size can be found by computer, however.

If

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 matrix its **determinant** is the number

$$\det \mathbf{A} = ad - bc.$$

(We are using a, b, c and d here because they are easier to write than A_{11}, A_{12}, A_{21} and A_{22}. Sometimes $\det \mathbf{A}$ is written as $|\mathbf{A}|$.) A 2×2 matrix $\mathbf{A}$ does not have an inverse if $\det \mathbf{A} = 0$, but if $\det \mathbf{A} \neq 0$ it does have an inverse and it is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \times \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This formula makes sense only when $\det \mathbf{A} \neq 0$ because otherwise the fraction $1/0$ appears. Let's check that this is the inverse:

$$\begin{aligned}
\mathbf{A}^{-1} \times \mathbf{A} &= \frac{1}{ad - bc} \times \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \frac{1}{ad - bc} \times \begin{bmatrix} d \times a + (-b) \times c & d \times b + (-b) \times d \\ (-c) \times a + a \times c & (-c) \times b + a \times d \end{bmatrix} \\
&= \frac{1}{ad - bc} \times \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \mathbf{I}
\end{aligned}$$

as required. You could also check that $\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$, though, as before, working out the product one way round is all that is necessary to check that we have the inverse of $\mathbf{A}$.

Examples

Let

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}.$$

Then

$$\det \mathbf{A} = (-1) \times 2 - (-1) \times 2 = 0$$

and

$$\det \mathbf{B} = 1 \times 4 - 2 \times 1 = 2,$$

so $\mathbf{B}$ has an inverse but $\mathbf{A}$ does not. The inverse of $\mathbf{B}$ is

$$\begin{aligned} \mathbf{B}^{-1} &= \frac{1}{2} \times \begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -1/2 & 1/2 \end{bmatrix}. \end{aligned}$$

Exercises

Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ -1 & 0 & 0 \\ 3 & -3 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -3 \\ 3 & 2 & -2 \\ -4 & 0 & -1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} -3 & -2 & 2 \\ 1 & 3 & 5 \\ -2 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix}.$$

1. What size are these matrices?

2. Write down the components A_{12}, A_{23} and A_{41}.

3. Calculate $\mathbf{A} + \mathbf{B}$.

4. Calculate $2\mathbf{C}$.

5. Calculate $\mathbf{A} + \mathbf{B} - 2\mathbf{C}$

6. Suppose

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 & 4 \\ 0 & 6 & 7 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 3 & 2 & 1 & -2 \\ 4 & 7 & -3 & -2 \end{bmatrix}$$

and $\mathbf{B}$ is matrix such tat

$$\mathbf{A} - \mathbf{B} = \mathbf{C}.$$

Find $\mathbf{B}$

In a study of mosquito populations the following figures were obtained:

	eggs	nos of wrigglers	tumblers	mosquitoes
1	2438	1104	358	210
Area 2	5068	2462	730	551
3	1438	982	412	289

A year later the numbers had changed to

$$\begin{bmatrix} 3700 & 1813 & 428 & 236 \\ 6238 & 3115 & 789 & 608 \\ 1649 & 1319 & 438 & 309 \end{bmatrix}$$

Let $\mathbf{A}$ be the matrix containing the first set of data, and $\mathbf{B}$ the matrix containing the second set of data.

7. What does the matrix $\mathbf{B} - \mathbf{A}$ represent? Calculate $\mathbf{B} - \mathbf{A}$.

Let

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}.$$

8. Write down the sizes of these three matrices.

9. Which of the matrix products $\mathbf{A}\times\mathbf{A}$, $\mathbf{A}\times\mathbf{B}$, $\mathbf{A}\times\mathbf{C}$, $\mathbf{B}\times\mathbf{A}$, $\mathbf{B}\times\mathbf{B}$, $\mathbf{B}\times\mathbf{C}$, $\mathbf{C}\times\mathbf{A}$, $\mathbf{C}\times\mathbf{B}$, and $\mathbf{C}\times\mathbf{C}$ are defined?

10. Do all the matrix multiplications in Exercise 9 that are defined.

A manufacturer makes two types of products I and II, at two plants X and Y. In the manufacturing process three types of pollutants – sulphur dioxide ($S0_2$), carbon monoxide (CO), and particle matter – are produced. The quantity of pollutants produced per day (in kg) in the manufacture of each product can be tabulated in a matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} SO_2 & \text{CO} & \text{particle} \end{array} \\ \begin{bmatrix} 150 & 50 & 100 \\ 200 & 25 & 150 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{product I} \\ \text{product II} \end{array}$$

To satisfy government regulations, the pollutants must be removed. The cost in dollars for removing each kg. of pollutant at each plant can be represented by the matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \text{plant} & \text{plant} \\ X & Y \end{array}$$

$$\mathbf{B} = \begin{bmatrix} 10 & 16 \\ 6 & 8 \\ 4 & 2 \end{bmatrix} \begin{array}{l} SO_2 \\ CO \\ \text{particle} \end{array}$$

11. Calculate **AB**. What information does it contain?

    Let **A** be the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}.$$

For any matrix **A** we define

$$\begin{aligned} \mathbf{A}^2 &= \mathbf{A} \times \mathbf{A} \\ \mathbf{A}^3 &= \mathbf{A} \times \mathbf{A} \times \mathbf{A} \end{aligned}$$

on so on.

12. Find $\mathbf{A}^2$.

13. Find $\mathbf{A}^3$ and $\mathbf{A}^4$.

14. Find $\mathbf{A}^{-1}$.

15. There are two ways to interpret $\mathbf{A}^{-2}$:

    (i) $\mathbf{A}^{-2} = (\mathbf{A}^{-1})^{2}$ (the square of the inverse of **A**,

    and

    (ii) $\mathbf{A}^{-2} = (\mathbf{A}^2)^{-1}$ (the inverse of the square of **A**).

    Check that these give the same answer.

13.8 Answers to Exercises

1. All are 4×3 matrices.

2. $A_{12} = 2$, $A_{23} = -3$ and $A_{41} = 3$.

3.

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -6 \\ 2 & 2 & -2 \\ -1 & -3 & -2 \end{bmatrix}$$

4.

$$2\mathbf{C} = \begin{bmatrix} -6 & -4 & 4 \\ 2 & 6 & 10 \\ -4 & -2 & 0 \\ -2 & -2 & -4 \end{bmatrix}$$

5.

$$\mathbf{A} + \mathbf{B} - 2\mathbf{C} = \begin{bmatrix} 7 & 6 & -1 \\ 1 & -6 & -16 \\ 6 & 4 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

6.

$$\mathbf{B} = \mathbf{A} - \mathbf{C} = \begin{bmatrix} -1 & -3 & 1 & 6 \\ -4 & -2 & 10 & 5 \end{bmatrix}$$

7. $\mathbf{B} - \mathbf{A} =$ the increase in mosquito populations over the year.

$$\mathbf{B} - \mathbf{A} = \begin{bmatrix} 1262 & 709 & 70 & 26 \\ 1170 & 653 & 59 & 57 \\ 211 & 337 & 26 & 20 \end{bmatrix}$$

8. $\mathbf{A}$ is 2×3, $\mathbf{B}$ is 2×2 and $\mathbf{C}$ is 3×1.

9.The products $\mathbf{A} \times \mathbf{C}$, $\mathbf{B} \times \mathbf{A}$ and $\mathbf{B} \times \mathbf{B}$ are defined.

10.

$$\mathbf{A} \times \mathbf{C} = \begin{bmatrix} 8 \\ -7 \end{bmatrix}$$

$$\mathbf{B} \times \mathbf{A} = \begin{bmatrix} -2 & 7 & 2 \\ 2 & -10 & -3 \end{bmatrix}$$

$$\mathbf{B}\times\mathbf{B} = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$$

11.

$$\mathbf{AB} = \begin{bmatrix} 2200 & 3000 \\ 2750 & 3700 \end{bmatrix}$$

this is the cost of removing pollutants from products at the plants. The rows correspond to the products, the columns the plants.

12.

$$\mathbf{A}^2 = \begin{bmatrix} 1 & -6 \\ 0 & 4 \end{bmatrix}$$

13.

$$\mathbf{A}^3 = \begin{bmatrix} 1 & -14 \\ 0 & 8 \end{bmatrix}$$

$$\mathbf{A}^4 = \begin{bmatrix} 1 & -30 \\ 0 & 16 \end{bmatrix}$$

14.

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

15.

$$\text{Both} = \begin{bmatrix} 1 & 1\frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}$$