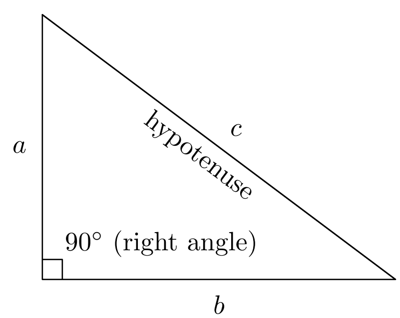
12 Trigonometry

From its Greek language origin, *trigonometry* deals with the measure of triangles. It is a very ancient and very important part of Science, with early applications in astronomy and navigation.

12.1 Pythagoras' Theorem

This was discovered, we are told, by the Greek, Pythagoras, or one of his philosophical and mathematical disciples in the sixth century B.C., but special cases of it were known many centuries earlier, to the Babylonians, for example. It is possibly the single most important theorem in Mathematics. Over 370 different proofs of this famous theorem are currently know.

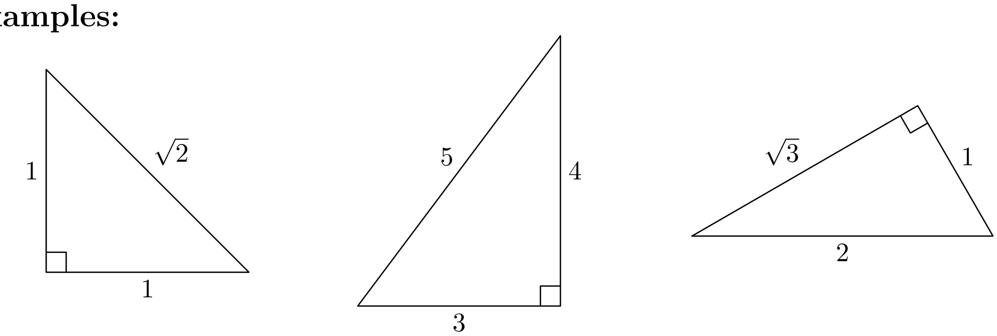
Pythagoras' Theorem: In the diagram



$$\boxed{c^2 = a^2 + b^2}$$

Pythagoras' Theorem

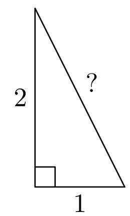
Examples:



Exercises 12.1:

Find the length of the side of the triangle indicated:

(i)



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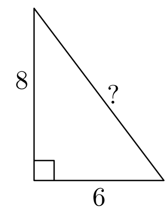

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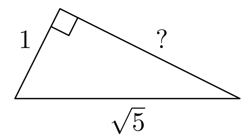

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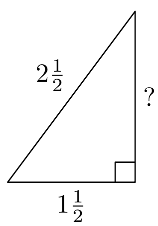

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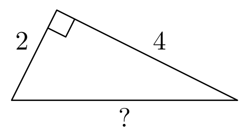

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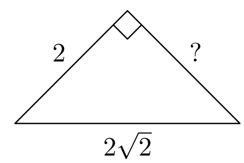

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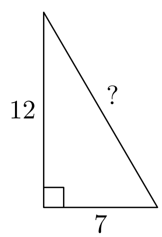

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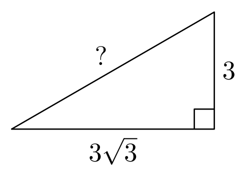
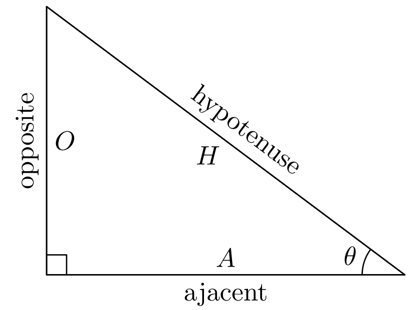


12.2 Trigonometrical Ratios



Let θ (Greek "theta") be an angle in a right-angled triangle. Represent the **hypotenuse** of the triangle, and the sides **opposite** and **adjacent** to θ, by H, O, A respectively. Then we define the following *trigonometrical ratios* (pronounce *sin* as "sine":

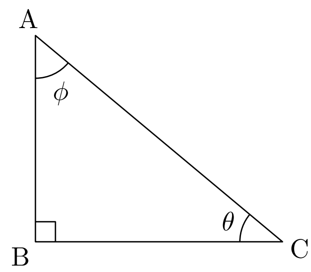
$$\begin{aligned} \text{the sine ratio: } \sin\,\theta &= \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{0}{H} \\ \text{the cosine ratio: } \cos\,\theta &= \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{A}{H} \\ \text{the tangent ration: } \tan\,\theta &= \frac{\text{opposite side}}{\text{adjacent side}} = \frac{0}{A} \end{aligned}$$

Thus $\boxed{\tan\theta = \dfrac{\sin\theta}{\cos\theta}} = \dfrac{\frac{0}{H}}{\frac{A}{H}} = \dfrac{0}{H} \times \dfrac{H}{A} = \dfrac{0}{A}$

[A *ratio* (which can be written as a fraction) compares the sizes of two quantities of the same kind. The trigonometrical ratios compare the lengths (i.e., the number of units of measurement) of two line-segments. Rational numbers (see Topic 3, Section 1) are expressible as ratios. Irrational numbers e.g. $\sqrt{3}$ (see Topic 9, Section 1) cannot be expressed as ratios of integers.]

There exist 3 other trigonometrical ratios which are the reciprocals of these, viz., $\dfrac{1}{\sin\theta} = \texttt{cosec}\ \theta$, $\dfrac{1}{\cos\theta} = \sec\theta$, $\dfrac{1}{\tan\theta} = \cot\theta$, but they don't concern us here. The names of the ratios have long historical associations.

It must be emphasized that the opposite side and adjacent side depend on the angle chosen. For example, with θ and ϕ (Greek "phi") as shown in the triangle ABC we have

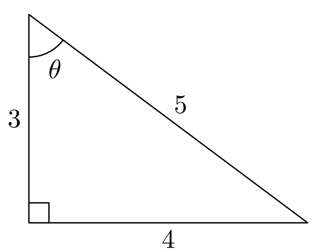


$$\begin{aligned} \sin\theta &= \frac{AB}{AC} \quad \sin\phi = \frac{BC}{AC} \quad (= \cos\theta) \\ \cos\theta &= \frac{BC}{AC} \quad \cos\phi = \frac{AB}{AC} \quad (= \sin\theta) \\ \tan\theta &= \frac{AB}{BC} \quad \tan\phi = \frac{BC}{AB} \quad (= \cot\theta). \end{aligned}$$

Though trigonometry deals with angles $> 90^\circ$ and angles $< 0^\circ$, we will restrict ourselves to angles θ for which $0^\circ \leq \theta \leq 90^\circ$.

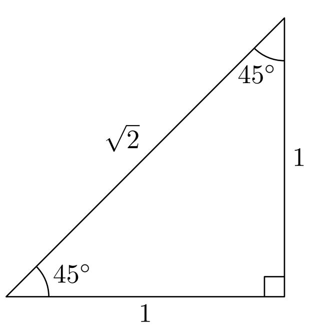
Examples:

(i)



$$\begin{aligned} \sin\theta &= \frac{4}{5} = 0.8 \\ \cos\theta &= \frac{3}{5} = 0.6 \\ \tan\theta &= \frac{4}{3} = 1.333\ldots \end{aligned}$$

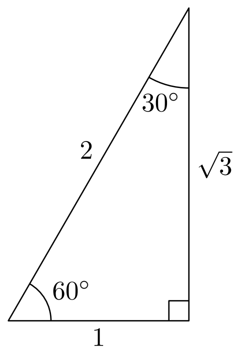
(ii)



$\boxed{45^\circ}$

$$\boxed{\begin{aligned} \sin 45^\circ &= \frac{1}{\sqrt{2}} = 0.7071\ldots \\ \cos 45^\circ &= \frac{1}{\sqrt{2}} = 0.7071\ldots \\ &= \sin 45^\circ \\ \tan 45^\circ &= \frac{1}{1} = 1 \end{aligned}}$$

(iii)



$\boxed{30^\circ, 60^\circ}$

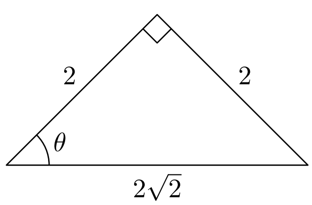
$$\boxed{\begin{aligned}
\sin 30^\circ &= \frac{1}{2} = 0.5 = \cos 60^\circ \\
\sin 60^\circ &= \frac{\sqrt{3}}{2} = 0.866\cdots = \cos 30^\circ \\
\tan 30^\circ &= \frac{1}{\sqrt{3}} = 0.5774\ldots \\
\tan 60^\circ &= \sqrt{3} = 1.7321\ldots
\end{aligned}}$$

The trigonometrical ratios of the angles 30°, 45°, 60° are used very often in Mathematics. It is worthwhile remembering them, or at least remembering the triangles from which the ratios may be obtained.

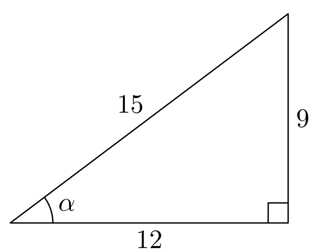
Exercises 12.2:

Find the $\sin, \cos$ and $\tan$ of the angle marked:

(i)



(ii)



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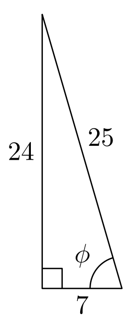

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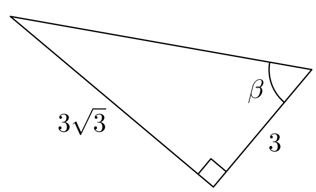
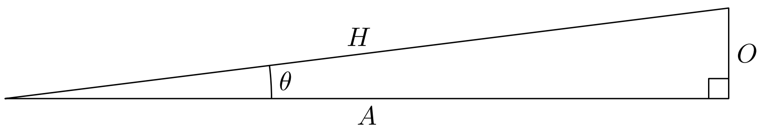


α, β are Greek "alpha" and "beta" respectively.

12.3 Angles $0°, 90°$

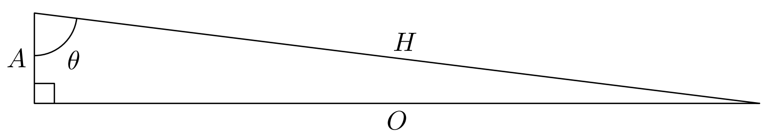
$\boxed{0°}$



In the diagram θ is very small. Imagine θ to get smaller and smaller so that 0 (opposite side) approaches zero and the hypotenuse (H) approaches the adjacent side (A). Ultimately,

$$\boxed{\begin{aligned} \sin 0° &= 0 \\ \cos 0° &= 1 \\ \tan 0° &= 0 \end{aligned}}$$

$\boxed{90°}$



In this diagram θ is large (but $< 90°$). Imagine θ to get closer and closer to $90°$ so that the opposite side (0) and the hypotenuse (H) become parallel and of "finite" length. Ultimately,

$$\boxed{\begin{aligned} \sin 90° &= 1 \\ \cos 90° &= 0 \\ \tan 90° &= \text{has no meaning } \left(= \frac{1}{0}\right) \end{aligned}}$$

12.4 More Advanced Results (for Reference)

It can be proved that for any angle θ

$$\boxed{\sin^2\theta + \cos^2\theta = 1} \qquad \ldots(\alpha)$$

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta} \qquad \ldots(\beta)$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta} \qquad \ldots(\gamma)$$

γ is the Greek letter "gamma".

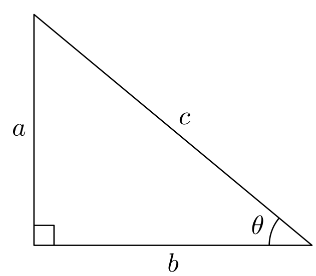
Exercises 12.3:

(i) From the above

(a) show that $\cos 2\theta \; = \; 2\cos^2\theta - 1 \; = \; 1 - 2\sin^2\theta$

(b) Find a formula for $\tan 2\theta \left(= \frac{\sin 2\theta}{\cos 2\theta}\right)$ in terms of $\tan\theta$

(ii) In the figure, $\sin\theta = \frac{a}{c}$, $\cos\theta = \frac{b}{c}$.



Hence prove (α).

(iii) Show that

(a) $\tan^2\theta + 1 = \sec^2\theta$

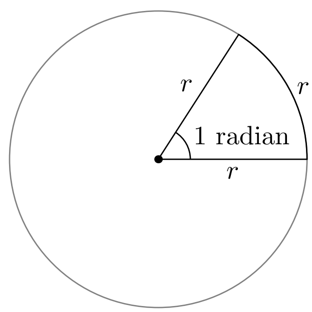
(b) $1 + \cot^2\theta = \texttt{cosec}^2\theta$

Radian measure

Usually in higher Mathematics (e.g., in Calculus), θ is measured in *radians* rather than *degrees* (which are likely to be used in practical situations, e.g., in carpentry and surveying).

Definition:

$$\begin{aligned} 2\pi \text{ radians} &= 360^\circ \\ 1 \text{ radian} &= \frac{360^\circ}{2\pi} \\ &= \frac{180^\circ}{\pi} = \frac{180^\circ}{3.14\ldots} \\ &= 57^\circ \quad \text{(approx.)} \end{aligned}$$



Thus, in radians we have: $\underbrace{90^\circ = \frac{\pi}{2}}_{\text{right angle}}$, $60^\circ = \frac{\pi}{3}$, $45^\circ = \frac{\pi}{4}$, $30^\circ = \frac{\pi}{6}$, $0^\circ = 0$.

Exercises 12.4:

(i) Express the following angles in radian measure:

(a) 180° (b) 18° (c) 36° (d) 135° (e) 150°

(ii) Express the following angles in degrees:

(a) π (b) $\frac{\pi}{10}$ (c) $\frac{\pi}{8}$ (d) $\frac{2\pi}{3}$ (e) $\frac{\pi}{12}$ (radians)

12.5 Answers to Exercises

12.1:

(i) $\sqrt{5}$ (iii) 2 (v) $\sqrt{20} = 2\sqrt{5}$ (vii) 13

(ii) 10 (iv) 2 (vi) 2 (viii) 6

12.2:

(i) $\theta = 45°$ so the answers are in Example (ii) above;

(ii)
$$\begin{aligned} \sin\alpha &= \frac{9}{15} = \frac{3}{5} = 0.6, \qquad \cos\alpha = \frac{12}{15} = \frac{4}{5} = 0.8, \\ \tan\alpha &= \frac{9}{12} = \frac{3}{4} = 0.75; \end{aligned}$$

(iii)
$$\begin{aligned} \sin\phi &= \frac{24}{25} = 0.96, \ \cos\phi = \frac{7}{25} = 0.28, \\ \tan\phi &= \frac{24}{7} = 3.4285\ldots; \end{aligned}$$

(iv) hypotenuse = 6 so the sides of the triangle are all 3 times the sides of the triangle in Example (iii). Therefore the other angles of the right-angled triangle are 30° and 60°. So the trigonometrical ratios are as in Example (iii) where $\beta = 60°$.

12.3:

(i) (a) Use results $(\alpha), (\gamma)$;

(b) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ from $(\beta), (\gamma)$

(ii)

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} \ \text{by Pythagoras' Theorem (12 Section 1)} \\ &= 1 \end{aligned}$$

(iii) (a) Divide (α) by $\cos^2\theta$ (b) Divide (α) by $\sin^2\theta$.

12.4:

(i) (a) π (b) $\frac{\pi}{10}$ (c) $\frac{\pi}{5}$ (d) $\frac{3\pi}{4}$ (e) $\frac{5\pi}{6}$

(ii) (a) 180° (b) 18° (c) $22\frac{1}{2}^\circ$ (d) 120° (e) 15°