11 Logarithms

Here we are concerned only with the basic theory of logarithms, not with computations using log tables.

Our main results depend on the main results in Topic 4.

11.1 Concept of a Logarithm

If $p = q^r$ we may also express this as *the* **logarithm** *of* p *to* **base** q *is* r. We write:

$\log_q p = r$

i.e. $\boxed{p = q^r \leftrightarrow log_q p = r}$

Examples:

(a) $8 = 2^3 \rightarrow \log_2 8 = 3$

(b) $9 = 3^2 \rightarrow log_3 9 = 2$

[$\rightarrow$ means "gives"]

Exercises 11.1

(i) $16 = 2^4 \rightarrow$?

(ii) $27 = 3^3 \rightarrow$?

(iii) $25 = 5^2 \rightarrow$?

(iv) $100 = 10^2 \rightarrow$?

Exercises 11.2

(i) $\log_5 25 = ?$

(ii) $\log_? 25 = 2$

(iii) $\log_3 81 = ?$

(iv) $\log_2 ? = 5$

(v) $\log_? 16 = 2$

(vi) $\log_4 64 = ?$

(vii) $\log_? 125 = 3$

(viii) $\log_7 ? = 2$

(ix) $\log_? 10 = 1$

(x) $\log_{10} ? = 5$

11.2 Basic Logarithm Theory

For any positive number a

$$\boxed{\log_a a = 1} \qquad \text{since } a^1 = a \;\; \text{(Topic 2, Section 3)}$$

$$\boxed{\log_a 1 = 0} \qquad \text{since } a^0 = 1 \;\; \text{(Topic 1, Section 6)}$$

If

$$\begin{cases} u = a^m \text{ then } \log_a u = m & \ldots \text{(i)} \\ v = a^n \text{ then } \log_a v = n & \ldots \text{(ii)} \end{cases}$$

$$\therefore\ uv = a^{m+n} \text{ i.e. } \log_a uv = m + n \quad \ldots \text{(iii)} \;\; \text{using Topic 4, Section 3}$$

$\therefore$ (i), (ii), (iii) give

$$\boxed{\log_a uv = \log_a u + \log_a v}$$

i.e. *the logarithm of a product = the sum of the logarithms* (all to the same base).

Similarly, from Topic 4, Section 4, we have

$$\boxed{\log_a\left(\frac{u}{v}\right) = \log_a u - \log_a v}$$

i.e. *the logarithm of a quotient = the difference of the logarithms* (same base).

$$\begin{array}{rcll} \text{Next, let } w & = & u^k = (a^m)^k & \text{from (i)} \\ & = & a^{mk} & \text{by Topic 4, Section 5} \\ \therefore \quad \log_a w & = & km & \text{by Section 1 above} \\ & = & k \log_a u & \text{from (i)} \end{array}$$

i.e.

$$\boxed{\log_a u^k = k \log_a u}$$

i.e. *the logarithm of a number with index k is k times the logarithm of the number* (same base).

Further, suppose

$$\begin{array}{rclcl} & & a & = & b^c \quad \text{i.e. } \log_b a = c \ \ldots \ \text{(iv)} \\ \therefore \quad a^{\frac{1}{c}} & = & \underbrace{(b^c)^{\frac{1}{c}} = b^{c.\frac{1}{c}}}_{\text{Topic 4, Section 5}} & = & b \quad \text{i.e. } \log_a b = \dfrac{1}{c} \ \ldots \ \text{(v)} \end{array}$$

(iv), (v) give $\log_b a \times \log_a b = c \times \frac{1}{c} = 1$

i.e.

$$\boxed{\log_b a \ \ \log_a b = 1}$$

i.e.

$$\log_a b = \tfrac{1}{\log_b a}$$

The most widely used bases are 10 and $\underbrace{e = 2.718\ \ldots}_{\text{irrational}}$

From tables,

$$\left\{ \begin{array}{lcl} \log_{10} e & = & \log_{10} 2.718.. = 0.4343\ldots \\ \log_e 10 & = & \frac{1}{\log_{10} e} = \frac{1}{0.4343..} = 2.3\ldots \end{array} \right.$$

11.3 Answers to Exercises

11.1:

(i) $\log_2 16 = 4$	(ii) $\log_3 27 = 3$	(iii) $\log_5 25 = 2$	(iv) $\log_{10} 100 = 2$

11.2:

(i) 2	(iv) 32	(vii) 5	(x) 100,000
(ii) 5	(v) 4	(viii) 49	
(iii) 4	(vi) 3	(ix) 10	