9 Inequalities

9.1 Real Numbers

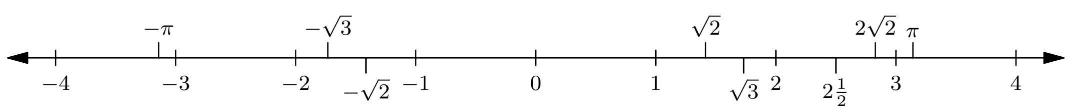
Numbers such as $\sqrt{2} = 1.4142\ldots, \sqrt{3} = 1.73\cdots, \pi = 3.14\cdots, e = 2.718\cdots$, which cannot be represented by a fraction (but are necessarily given by a non-terminating, non-repeating decimal) are called *irrational numbers*, discovered by Pythagoras. [π is Greek “pi”.]

There is an infinite number of irrational numbers.

Irrational numbers and rational numbers (Topic 3, Section 1) form the system of *real numbers.*

The rational number line (Topic 3, Section 1) can be extended to account for irrational numbers. There are now **no gaps** at all on the real number line.

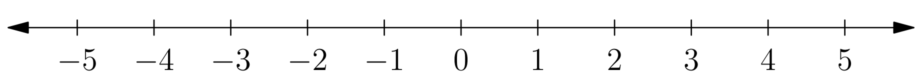
Every point on the real number line corresponds to a real number, and every real number can be represented by a point on the real number line.



Real Number Line

9.2 Inequalities

Inequalities (i.e., expressions involving the signs $>$ (”is greater than”) or $<$ (”is less than”) are readily visualised geometrically on the real number line (e.g., $4 > 3, 1\frac{1}{2} < 2$). The aspects featured in the following two examples are important.



Example:

$$\underbrace{-3 < -1 \text{ is the same as } -1 > -3}_{\text{change of inequality signs}}$$

Example:

$$\underbrace{4 > -1 \quad \text{is the same as} \quad -4 < 1}_{\substack{\text{multiplying both sides of an} \\ \text{inequality by -1 reverses} \\ \text{the inequality sign.}}}$$

When multiplying, or dividing, an inequality by a negative number, the inequality sign **must** change.

Exercises 9.1:

(i) (a) $3 \,\square\, -1$

(b) $-2 \,\square\, 5$

(c) $-2 \,\square\, -3$

(d) $-3 \,\square\, 2$ is the same as $-2 \,\square\, 3$

(ii) Which is greater:

(a) 2^3 or $(-3)^2$;

(b) 2^{-3} or $(-3)^{-2}$;

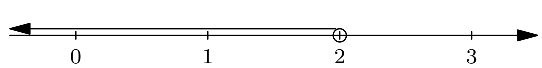
(c) $\left(-\frac{1}{2}\right)^3$ or $\left(-\frac{1}{3}\right)^2$?

9.3 Solution of Inequalities

Example: Solve $x - 7 < -5$.

Solution:

$$\begin{aligned} x - 7 + 7 &< -5 + 7 \\ x &< 2 \end{aligned}$$



$\therefore$ the solution is the infinite number of real numbers less than 2.

When solving linear inequalities, apply the techniques learnt for solving linear equation (as in the above Example) **except** in one important instance. This exception occurs when you need to multiply, or divide, by a negative number.

Consider

(a)

$$\begin{aligned} 3(x+4) &< 5 \\ 3x+12 &< 5 \\ 3x &< -7 \\ x &< -2\frac{1}{3} \quad \text{dividing by 3} \end{aligned}$$

(b)

$$\begin{aligned} -3(x+6) &> 15 \\ -3x-18 &> 15 \\ -3x &> 33 \\ x &< -11 \quad \text{dividing by -3} \quad \text{(inequality sign changes)} \end{aligned}$$

Exercises 9.2:

Solve

(i) $x + 6 > 2$

(ii) $x - 3 > -1$

(iii) $2x + 1 < x + 4$

(iv) $1 - 2x > x + 2$

(v) $3x + 6 > 2(x + 3)$

A problem on inequalities can readily be extended to include equality. For instance, consider $x - 7 \leq -5$ (i.e. $x - 7$ is **less than or equal** to -5). From the solution to the Example above, $x \leq 2$, i.e., the solution is the infinite number of real numbers less than or equal to 2, i.e., all (real) numbers represented by points to the left of, or at, 2.

9.4 Answers to Exercises

9.1:

(i) (a) $>$ (b) $<$ (c) $>$ (d) $<, <$

(ii) (a) $(-3)^2$ (b) 2^{-3} (c) $\left(-\frac{1}{3}\right)^2$

9.2:

(i) $x > -4$

(ii) $x > 2$

(iii) $x < 3$

(iv) $x < -\frac{1}{3}$

(v) $x > 0$