7 Quadratic Expressions

A *quadratic* expression (Latin *quadratus* $\equiv$ "squared") is an expression involving a squared term, e.g., $x^2 + 1$, or a product term, e.g., $3xy - 2x + 1$. (A linear expression such as $x + 1$ is obviously non-quadratic.)

7.1 Expansion (Multiplication)

We have seen in Topic 2, Section 3 how to multiply, or expand, simple expressions involving one linear factor such as $3(2x - 5)$ i.e. $3(2x - 5) = 3(2x) - 3 \times 5 = 6x - 15$.

Now we wish to multiply (expand) two linear expressions - the product is naturally a quadratic expression. Always simplify the quadratic expression, if possible.

Example:

$$\begin{aligned}
(2x-5)(x+2) &= 2x(x+2) - 5(x+2) \\
&= 2x \times x + 2x \times 2 - 5x - 10 \\
&= 2x^2 + 4x - 5x - 10 \\
&= 2x^2 - x - 10 \qquad \text{on simplification.}
\end{aligned}$$

The order of the factors is unimportant, i.e.

$$(2x-5)(x+2) = (x+2)(2x-5)$$

Exercises 7.1: Multiply (expand)

(i) $(x+3)(x-4)$

(ii) $(2x-3)(3x-2)$

(iii) $(3x-1)(2-x)$

(iv) $(x+1)(y+2)$

(v) $(2x-3)^2$

(vi) $(x+1)(x-1)$

(vii) $(7x+4)(7x-4)$

(viii) $(x+a)(x+b)$

(ix) $(x+a)(x-a)$

(x) $(x-a)^2$

(xi) $(2x-3y)(x+4y)$

(xii) $(2x-5t)^2$

Note: In (ix), the factors $x+a, x-a$ differ only in the sign in front of a, leading to the *difference of squares* x^2-a^2. (The term in ax is $+ax-ax=0$, i.e., there is **no** term in ax.) In (x), the square of $x-a$, namely $(x-a)(x-a)$, **will** have a term in ax, i.e., $-ax-ax(=-2ax)$. The squared terms are $x \cdot x = x^2$ and $(-a)(-a)=+a^2$.

7.2 Factorizing

In Section 1 above, we were given the factors and we had to find the equivalent unfactored expression.

Now we are given an unfactored expression and we are asked to find the factors, i.e., to "factorize" the expression. Guesswork, to some extent, is involved in this technique, until you become proficient at factorizing.

Example: Factorize $x^2 - x - 12$

Solution: we look for an answer of the kind

$$(x+a)(x+b) \; [= x^2 + (a+b)x + ab \text{ by Exercise 7.1, (viii) above}]$$

This means $a + b = -1, ab = -12$ in our Example.

Solving by "trial-and-error" gives $a = 3, b = -4$ (or $a = -4, b = 3$)

$$\therefore \; x^2 - x - 12 = (x+3)(x-4).$$

The order of the factors in the answer is unimportant. Check back to Section 1 Exercise 7.1 (i) for the reverse problem (expansion).

When factorizing a more complicated quadratic such as $6x^2 + 7x - 20$, we apply the same general principles, though the problem is harder. We have

$$6x^2 + 7x - 20 = (2x+5)(3x-4) \qquad \begin{array}{c} 2x + 5 \\ \times \\ 3x - 4 \end{array}$$

Exercises 7.2:

Factorize

(i) $x^2 + 4x + 3$

(ii) $x^2 - 7x + 12$

(iii) $x^2 - 2x - 8$

(iv) $x^2 + x - 2$

(v) $6x^2 - 7x - 20$

(vi) $12x^2 - x - 6$

(vii) $x^2 - 25$

(viii) $x^2 - 6x + 9$

(ix) $x^2 + 2ax + a^2$

(x) $4x^2 - 4x + 1$

7.3 Functional Notation

In everyday language, the statement

"the number of hours of daylight **depends on** the time of year"

could also be expressed as

"the number of hours of daylight **is a function of** the time of year".

In Mathematics, the word "function" is used with essentially the same meaning, though somewhat idealized.

For example, in $y = 2x^2 - 3x + 5$, to a particular value of x there corresponds a **unique** value of y (e.g. $x = 1$ gives $y = 4$).

We then say that $2x^2 - 3x + 5$ is a **function** of x and, for convenience, replacing y by $f(x)$, write

$$f(x) = 2x^2 - 3x + 5.$$

Read the notation $f(x)$ as "f of x" where f is a symbol for the particular function of x under consideration.

Note: $f(x)$ does **not** mean $f \times x$, i.e. the brackets do not signify multiplication. Any symbol other than f can be used for a function. The **value** of $f(x)$ when $x = 3$ say is written $f(3)$.

Example: Find the value of

(i) $f(x) = 3x - 4$ when $x = \frac{1}{2}$

(ii) $f(x) = 2x^2 + 4x - 7$ when $x = 3$.

Solutions:

(i)

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 3 \times \frac{1}{2} - 4 \\ &= 1\frac{1}{2} - 4 \\ &= -2\frac{1}{2} \end{aligned}$$

(ii)

$$\begin{aligned} f(3) &= 2 \times (3 \times 3) + (4 \times 3) - 7 \\ &= 18 + 12 - 7 \\ &= 23 \end{aligned}$$

Test for a Factor of $f(x)$: If $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

Example:

$$\begin{array}{rrcl}
\text{Let} & f(x) & = & x^2 - 7x + 12 \\
\text{Then} & f(3) & = & 3 \times 3 - 7 \times 3 + 12 \\
& & = & 9 - 21 + 12 \\
& & = & 0 \\
\text{and} & f(4) & = & 4 \times 4 - 7 \times 4 + 12 \\
& & = & 16 - 28 + 12 \\
& & = & 0 \\
\therefore & x^2 - 7x + 12 & = & (x-3)(x-4) \quad \text{(see Exercise 7.2 (ii), Section 2 above)}
\end{array}$$

Exercises 7.3:

(i) Find the value of $f(x) = 2x^2 + 4x - 9$ when

(a) $x = 2$

(b) $x = 0$

(c) $x = \frac{1}{2}$

(d) $x = -1$

(ii) Test whether $x - 2$ is a factor of

(a) $g(x) = 2x^2 - 3x - 2$

(b) $h(x) = 3x^2 + x - 10$

Note: A *cubic* expression has an index 3 in the variable, e.g. $x^3 + x^2 - 20x$ is a cubic expression (and it happens to factorize, namely,

$$x^3 + x^2 - 20x = x(\underbrace{x^2 + x - 20}_{\text{quadratic}}) = x(x+5)(x-4)).$$

It is worth noting, as you can readily check, that

$$\begin{cases} x^3 - a^3 & = & (x-a)(x^2 + ax + a^2) \\ x^3 + a^3 & = & (x+a)(x^2 - ax + a^2) \end{cases}$$

$$\text{e.g.}\begin{cases} x^3 - 1 &= (x-1)(x^2+x+1) \\ x^3+8 &= (x+2)(x^2-2x+4) \\ 8x^3-27 &= (2x-3)(4x^2+6x+9). \end{cases}$$

7.4 Solving a Quadratic Equation

Sometimes a quadratic equation has factors in the quadratic expression. In this case it is easy to solve the equation.

Example: Solve $x^2 - x - 12 = 0$

Solution:
$$\begin{aligned} \text{Now} \quad x^2 - x - 12 &= (x+3)(x-4) \quad \text{(See Topic 7, Section 2)} \\ \therefore \quad (x+3)(x-4) &= 0 \qquad \text{i.e.} \quad x+3=0 \text{ or } x-4=0 \\ \therefore \quad x &= -3, 4 \end{aligned}$$

i.e., there are **two** solutions of a quadratic equation. Checking gives $(-3)^2 - (-3) - 12 = 9 + 3 - 12 = 0$, and $4^2 - 4 - 12 = 16 - 4 - 12 = 0$, so the answer is correct.

Solving $x^2 - 4 = 0 = (x-2)(x+2)$, we get $x = 2, -2$ which is also written $x = \pm 2$. Or, *taking square roots of both sides* of $x^2 = 4$ (i.e. $x^2 - 4 = 0$), we have $x = +\sqrt{4}, -\sqrt{4} = \pm\sqrt{4} = \pm 2$. Note that $2 \times 2 = 4$ and also $(-2) \times (-2) = 4$.

We now obtain a useful formula for solving any quadratic equation $ax^2 + bx + c = 0$.

An Important Formula: If $ax^2 + bx + c = 0$, then

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Proof: Multiply both sides of $ax^2 + bx + c = 0$ by $4a$ to get

$$\begin{aligned} 4a(ax^2 + bx + c) &= 4a \times 0 \\ 4a^2x^2 + 4abx + 4ac &= 0 \\ 4a^2x^2 + 4abx &= -4ac \end{aligned}$$

Add b^2 to both sides to turn the left-hand side into a perfect square

$$\begin{aligned} \text{i.e.} \qquad 4a^2x^2 + 4abx + b^2 &= b^2 - 4ac \\ (2ax + b)^2 &= b^2 - 4ac \end{aligned}$$

Take the square root of both sides. We have

$$\begin{aligned}
2ax+b &= \pm\sqrt{b^2-4ac} = \sqrt{b^2-4ac}, \sqrt{b^2-4ac} \\
2ax &= -b \pm \sqrt{b^2-4ac} \\
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}
\end{aligned}$$

Note: that **all** of $-b \pm \sqrt{b^2-4ac}$ is divided by $2a$.

Example: Solve $3x^2 - 9x + 5 = 0$

Solution: In the formula, substitute $a = 3, b = -9, c = 5$ to get

$$\begin{aligned}
x &= \frac{(-9) \pm \sqrt{(-9)^2 - 4 \times 3 \times 5}}{2 \times 3} \\
&= \frac{9 \pm \sqrt{81-60}}{6} \\
&= \frac{9 \pm \sqrt{21}}{6} \\
&= \frac{9+\sqrt{21}}{6}, \frac{9-\sqrt{21}}{6}
\end{aligned}$$

Of course, if we apply the formula to a quadratic expression which factorizes, the calculations will produce a perfect square under the square root sign $\sqrt{\ }$, and so $\sqrt{\ }$ will disappear from the answer. Thus, in the Example at the beginning of Section 4, in $x^2 - x - 12 = 0$ we have $a = 1, b = -1, c = -12$. The formula yields

$$\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-12)}}{2 \times 1} \\
&= \frac{1 \pm \sqrt{1+48}}{2} \\
&= \frac{1 \pm \sqrt{49}}{2} \\
&= \frac{1 \pm 7}{2} \\
&= \frac{8, -6}{2} \\
&= 4, -3 \qquad \text{as obtained in the Example.}
\end{aligned}$$

Exercises 7.4: Solve

(i) $2x^2 - 5x + 2 = 0$

(ii) $6x^2 - 7x - 3 = 0$

(iii) $x^2 - x - 1 = 0$

(iv) $2t^2 + 5t + 1 = 0$

7.5 Operations with Square Roots

Here we deal briefly with simple arithmetical operations involving square roots. Some representative examples are chosen.

Example: Simplify (a) $\sqrt{8}$ (b) $\sqrt{6}$

Solution:

(a) $\sqrt{8} = \sqrt{2 \times 2 \times 2} = \sqrt{2^2 \times 2} = 2\sqrt{2}$ since $\sqrt{2^2} = 2$

(b) $\sqrt{6} = \sqrt{2 \times 3} = \sqrt{2}\sqrt{3}$

i.e. the square root of a product is the product of the square roots.
So, generally, $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Similarly, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

Example: Simplify $\frac{1}{\sqrt{3}}$.

Solution: Multiply by $\frac{\sqrt{3}}{\sqrt{3}}(= 1)$

$$\begin{aligned} \therefore \quad \frac{1}{\sqrt{3}} &= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3} \qquad\qquad \text{since } (\sqrt{3})^2 = 3 \end{aligned}$$

This process is called *rationalising the denominator*, i.e. the denominator, $\sqrt{3}$, which is irrational, is replaced by a denominator (3) which is a rational number.

Exercises 7.5: Simplify

(i) $\sqrt{27}$

(ii) $\sqrt{32}$

(iii) $\sqrt{10}$

(iv) $\frac{1}{\sqrt{2}}$

(v) $\frac{2}{\sqrt{5}}$

(vi) $\frac{\sqrt{14}}{\sqrt{2}}$

(vii) $\frac{\sqrt{27}}{\sqrt{12}}$

Exercises 7.6: Solve

(i) $2x^2 - 4x - 3 = 0$

(ii) $3w^2 - 8w + 3 = 0$

Topic Revision Exercises 7.7:

(i) Expand $(2x + 3)(2 - 3x)$

(ii) Factorize $10x^2 + 7x - 12$

(iii) If $f(x) = 3 - 8x + 2x^2$ find the value of

(a) $f\left(\frac{1}{2}\right)$

(b) $f\left(-\frac{1}{2}\right)$

(iv) Test whether $x - 2$ is a factor of $3x^2 - 2x - 6$

(v) Simplify $\frac{\sqrt{48}}{\sqrt{36}}$

(vi) Solve

(a) $x^2 - 9 = 0$

(b) $3x^2 - 8x + 2 = 0$

(c) $4x^2 - 4x + 1 = 0$

7.6 Answers to Exercises

7.1:

(i) $x^2 - x - 12$

(ii) $6x^2 - 13x + 6$

(iii) $-3x^2 + 7x - 2$

(iv) $xy + 2x + y + 2$ (this is still a quadratic as it involves the product xy of two variables (x, y))

(v) $4x^2 - 12x + 9$

(vi) $x^2 - 1$

(vii) $49x^2 - 16$

(viii) $x^2 + (a + b)x + ab$

(ix) $x^2 - a^2$

(x) $x^2 - 2ax + a^2$

(xi) $2x^2 + 5xy - 12y^2$

(xii) $4x^2 - 20xt + 25t^2$

7.2:

(i) $(x + 1)(x + 3)$

(ii) $(x - 3)(x - 4)$

(iii) $(x + 2)(x - 4)$

(iv) $(x - 1)(x + 2)$

(v) $(3x + 4)(2x - 5)$

(vi) $(3x + 2)(4x - 3)$

(vii) $(x + 5)(x - 5)$

(viii) $(x - 3)^2$

(ix) $(x + a)^2$

(x) $(2x - 1)^2$

7.3:

(i) (a) $f(2) = 7$ (b) $f(0) = -9$ (c) $f\left(\frac{1}{2}\right) = -6\frac{1}{2}$ (d) $f(-1) = -11$

(ii) (a) $g(2) = 0$ so $x - 2$ is a factor, the other factor being $2x + 1$;
(b) no, it is not a factor since $h(2) = 4 \neq 0$.

7.4:

(i) $x = \frac{1}{2}, 2$

(ii) $x = 1\frac{1}{2}, -\frac{1}{3}$

(iii) $x = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

(iv) $t = \frac{-5+\sqrt{17}}{4}, \frac{-5-\sqrt{17}}{4}$

7.5:

(i) $3\sqrt{3}$

(ii) $4\sqrt{2}$

(iii) $\sqrt{2}\sqrt{5}$

(iv) $\frac{\sqrt{2}}{2}$

(v) $\frac{2}{5}\sqrt{5}$

(vi) $\sqrt{7}$

(vii) $\frac{3}{2}$

7.6:

(i) $x = 1 + \frac{\sqrt{10}}{2}, 1 - \frac{\sqrt{10}}{2}$

(ii) $w = \frac{4+\sqrt{7}}{3}, \frac{4-\sqrt{7}}{3}$

7.7:

(i) $-6x^2 - 5x + 6$

(ii) $(2x+3)(5x-4)$

(iii) (a) $-\frac{1}{2}$ (b) $7\frac{1}{2}$

(iv) If $g(x) = 3x^2 - 2x - 6$, then $g(2) = 2(\neq 0)$ so $x - 2$ is **not** a factor of $g(x)$

(v) $\frac{2\sqrt{3}}{3}$

(vi) (a) $x = 3, -3$ (b) $x = \frac{4\pm\sqrt{10}}{3}$
(c) $x = \frac{1}{2}, \frac{1}{2}$ (a repeated solution as $4x^2 - 4x + 1 = (2x-1)^2$, i.e. a square.)