

3 Fractions

3.1 Extension of the Number Line

A *fraction* is a number of the form $\pm \frac{a}{b}$ where a and b are integers. In a fraction, such as $\frac{3}{4}$,

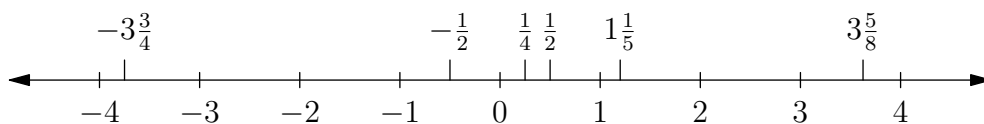
3 (top) is called the *numerator*

4 (bottom) is called the *denominator*.

Integers and fractions together form the system of *rational numbers*, (i.e., numbers expressible as *ratios* - see Topic 12, Section 2). Observe that an integer may be written in fractional form, e.g.

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{10}{5} = \dots$$

Our number line for integers (see Topic 1, Section 1) can be extended to account for fractions, i.e., some of the geometrical gaps are now filled by points representing fractions.



Note: There are still empty places on this line to be filled by points representing *irrational numbers*, i.e. numbers which are not rational e.g. $\sqrt{2}$, π (see Topic 9, Section 1).

3.2 Arithmetic of Fractions

Examples: To calculate the answer to the sum of $\frac{1}{3} + \frac{2}{5}$ we must first find the lowest denominator that is common to both fractions.

$$(a) \frac{1}{3} + \frac{2}{5} = \left(\frac{1}{3} \times \frac{5}{5}\right) + \left(\frac{2}{5} \times \frac{3}{3}\right) = \frac{1 \times 5 + 2 \times 3}{3 \times 5} = \frac{5+6}{15} = \frac{11}{15}$$

Note: $\boxed{\frac{1}{3}} + \boxed{\frac{2}{5}} = \frac{1}{3} \times \frac{5}{5} + \frac{2}{5} \times \frac{3}{3} = \boxed{\frac{5}{15}} + \boxed{\frac{6}{15}} = \frac{5+6}{15} = \frac{11}{15}$

This explains what is really happening in the “rule-of-thumb” method used in Example (a).

Always simplify your answer, e.g. $\frac{3}{5} \times \frac{10}{9} = \frac{2}{3}$.

$$(b) \frac{1}{3} - \frac{2}{5} = \frac{5-6}{15} = -\frac{1}{15}$$

$$(c) \frac{1}{3} \times \frac{2}{5} = \frac{1 \times 2}{3 \times 5} = \frac{2}{15} \quad (\text{Note: } \frac{1}{3} \text{ of } \frac{2}{5} \text{ means } \frac{1}{3} \times \frac{2}{5})$$

$$(d) \frac{1}{3} \div \frac{2}{5} = \frac{1}{3} \times \frac{5}{2} = \frac{5}{6} \quad (\text{To divide by } \frac{2}{5}, \text{ multiply by } \frac{5}{2})$$

In Example (d), the fraction may also be written as

$$\frac{\frac{1}{3}}{\frac{2}{5}} \quad \text{or} \quad \frac{1}{3} / \frac{2}{5}.$$

Care should be taken when using / e.g. $1/2 \times 3 = 1/(2 \times 3) (= \frac{1}{6})$ could be carelessly written as $1/2 \times 3$ leading to the interpretation $\frac{1}{2} \times 3 (= 1\frac{1}{2})$.

Exercises 3.1:

(i) $\frac{1}{4} + \frac{1}{2}$

(ii) $\frac{1}{2} + \frac{1}{3}$

(iii) $\frac{1}{2} - \frac{1}{3}$

(iv) $\frac{1}{4} + \frac{1}{4}$

(v) $\frac{3}{5} - \frac{4}{7}$

(iv) $\frac{6}{9} - \frac{8}{12}$

(vii) $\frac{2}{3} \times \frac{3}{7}$

(viii) $-\frac{3}{4} \times \frac{2}{5}$

(ix) $-\frac{5}{4} \times -\frac{8}{15}$

(x) $\frac{2}{3} \div \frac{4}{7}$

(xi) $\frac{2}{3} \div (-\frac{8}{9})$

(xii) $2^{-3} - 3^{-2}$

(xiii) $(\frac{1}{2})^{-2} - (\frac{1}{3})^{-3}$

(xiv) $\frac{2}{7} \times 0$

(xv) $0 \div \frac{3}{4}$

(xvi) $\frac{3}{4} \div 0$

(xvii) $\frac{2}{5} \times (\frac{2}{5})^{-1}$

3.3 Percentages

“Percent” (symbolically %) means per hundred (from the Latin word *centum* \equiv “hundred”)

$$\begin{aligned} \text{e.g } 5\% &= \frac{5}{100} = \frac{1}{20} \quad (\text{in its simplest form}) \\ \frac{3}{10} &= \frac{30}{100} = 30\%. \end{aligned}$$

Note: 5% of 30 means $\frac{5}{100} \times 30 = 1\frac{1}{2}$

i.e., the preposition “of” in this context implies multiplication.

Exercises 3.2:

Express (i) - (v) as fractions in their simplest form:

(i) 10%

(ii) 75%

(iii) $33\frac{1}{3}\%$

(iv) $2\frac{1}{2}\%$

(v) 14%.

Express (vi) - (viii) as percentages:

(vi) $\frac{1}{5}$

(vii) $\frac{3}{8}$

(viii) $\frac{9}{20}$.

Find:

(ix) $7\frac{1}{2}\%$ of 80

(x) 12% of \$25

3.4 Decimals

Because of the common usage of decimals in our day-to-day money transactions, we will not develop any manipulative procedure with them. It is assumed that you can carry out fairly simple calculations involving decimals.

However, observe that fractions in decimal form are of two kinds:

- terminating e.g. $\frac{1}{2} = 0.5$, $\frac{5}{8} = 0.625$;
- non-terminating, but repeating e.g.

$$\begin{aligned}\frac{1}{3} &= 0.333\dots, \\ \frac{4}{7} &= 0.5714285714285\dots\end{aligned}$$

The word “decimal” comes from the Latin word *deimus* \equiv “tenth”, i.e., our decimal system is based on 10:

$$\begin{aligned}10^{-1} &= \frac{1}{10} = 0.1, \\ 10^{-2} &= \frac{1}{10^2} = 0.01, \\ 10^{-3} &= \frac{1}{10^3} = 0.001, \\ 10^{-4} &= \frac{1}{10^4} = 0.0001 \text{ and so on.}\end{aligned}$$

Later (Topic 9, Section 1) you will notice that irrational numbers (e.g. $\sqrt{2}$) can be written as non-terminating, non-repeating decimals.

3.5 Harder fractions

Example:

$$\frac{\frac{1}{2} - \frac{1}{3}}{\frac{3}{4}}$$

Do this in two stages.

$$\begin{aligned} \text{Firstly: } \quad \frac{1}{2} - \frac{1}{3} &= \left(\frac{1}{2} \times \frac{3}{3}\right) - \left(\frac{1}{3} \times \frac{2}{2}\right) \\ &= \frac{3}{6} - \frac{2}{6} \\ &= \frac{1}{6} \\ \text{Secondly: } \quad \frac{1}{6} \div \frac{3}{4} &= \frac{1}{6} \times \frac{4}{3} \\ &= \frac{4}{18} \\ &= \frac{2}{9} \end{aligned}$$

When a number such as $6\frac{3}{4}$ occurs it is often necessary to bring it to the ordinary fractional form: $\frac{27}{4}$ ($= \frac{6 \times 4}{4} + \frac{3}{4} = \frac{24+3}{4}$)

Examples:

$$\begin{aligned} \text{(a) } 6\frac{3}{4} - 4\frac{2}{3} &= \frac{27}{4} - \frac{14}{3} & \text{OR} & \quad 6\frac{3}{4} - 4\frac{2}{3} = (6 + \frac{3}{4}) - (4 + \frac{2}{3}) \\ &= \frac{81-56}{12} & & \quad = (6-4) + (\frac{3}{4} - \frac{2}{3}) \\ &= \frac{25}{12} & & \quad = 2 + \frac{9-8}{12} \\ &= 2\frac{1}{12} & & \quad = 2\frac{1}{12} \end{aligned}$$

(b)

$$\begin{aligned} 3\frac{1}{3} \div 1\frac{1}{4} &= \frac{10}{3} \div \frac{5}{4} \\ &= \frac{10}{3} \times \frac{4}{5} \\ &= \frac{8}{3} \\ &= 2\frac{2}{3} \end{aligned}$$

Exercises 3.3:

(i) $\left(\frac{1}{3} + \frac{3}{5}\right) \div \frac{7}{9}$

(ii) $7\frac{2}{3} - 4\frac{1}{2}$

(iii) $2\frac{3}{4} + 3\frac{1}{2}$

(iv) $2\frac{2}{3} \times 2\frac{1}{4}$

(v) $1\frac{5}{9} \div 1\frac{1}{6}$

(vi) $3\frac{2}{3} - 4\frac{1}{2}$

3.6 Algebra

Our algebraic examples are modelled on the corresponding arithmetical examples just completed in Section 2 above.

Examples:

(a)

$$\frac{x}{3} + \frac{2x}{5} = \frac{x}{3} \times \frac{5}{5} + \frac{2x}{5} \times \frac{3}{3} = \frac{5x}{15} + \frac{6x}{15} = \frac{5x + 6x}{15} = \frac{11x}{15}$$

(b)

$$\frac{x}{3} - \frac{2x}{5} = -\frac{x}{15}$$

(c)

$$\frac{x}{3} \times \frac{2x}{5} = \frac{2x^2}{15}$$

(d)

$$\frac{x}{3} \div \frac{2x}{5} = \frac{x}{3} \times \frac{5}{2x} = \frac{5x}{6x} = \frac{5}{6}$$

Exercises 3.4:

(i) $\frac{x}{2} + \frac{x}{2}$

(ii) $\frac{x}{2} + \frac{x}{3}$

(iii) $\frac{3x}{4} - \frac{2x}{3}$

(iv) $\frac{7x}{2} + \frac{8x}{5}$

(v) $\frac{5x}{4} \times \frac{8x}{15}$

(vi) $-\frac{x}{4} \div \frac{x}{2}$

(vii) $-\frac{x}{4} \div \frac{2}{x}$

(viii) $\frac{0}{2x}$

(ix) $\frac{2x}{0}$

(x) $\frac{5x}{7} - \frac{2x}{3}$

(xi) $\frac{2}{3x} \div \frac{4x}{9}$

(xii) $\frac{1}{x} + \frac{1}{x}$

(xiii) $\frac{1}{2x} + \frac{1}{2x}$

(xiv) $\frac{x}{2} - \frac{2}{x}$

(xv) $2x - \frac{5}{6}x + \frac{x}{3} - \frac{x}{2}$

3.7 Answers to Exercises

3.1:

- | | | | |
|---------------------|--|----------------------|--------------------|
| (i) $\frac{3}{4}$ | (vi) 0 | (xi) $-\frac{3}{4}$ | (xvi)) no meaning |
| (ii) $\frac{5}{6}$ | (vii) $\frac{2}{7}$ ($= \frac{6}{21}$) | (xii) $\frac{1}{72}$ | (xvii) 1 |
| (iii) $\frac{1}{6}$ | (viii) $-\frac{3}{10}$ | (xiii) -23 | |
| (iv) $\frac{1}{2}$ | (ix) $\frac{2}{3}$ | (xiv) 0 | |
| (v) $\frac{1}{35}$ | (x) ($= 1\frac{1}{6}$) | (xv) 0 | |

3.2:

- | | | | |
|---------------------|---------------------|-------------------------|---------|
| (i) $\frac{1}{10}$ | (iv) $\frac{1}{40}$ | (vii) $37\frac{1}{2}\%$ | (x) \$3 |
| (ii) $\frac{3}{4}$ | (v) $\frac{7}{50}$ | (viii) 45% | |
| (iii) $\frac{1}{3}$ | (vi) 20% | (ix) 6 | |

3.3:

- | | | |
|---------------------|----------------------|---------------------|
| (i) $1\frac{1}{5}$ | (iii) $6\frac{1}{4}$ | (v) $1\frac{1}{3}$ |
| (ii) $3\frac{1}{6}$ | (iv) 6 | (vi) $-\frac{5}{6}$ |

3.4:

- | | | | |
|-----------------------|------------------------|-----------------------|--------------------------|
| (i) x | (v) $\frac{2x^2}{3}$ | (ix) no meaning | (xiii) $\frac{1}{x}$ |
| (ii) $\frac{5x}{6}$ | (vi) $-\frac{1}{2}$ | (x) $\frac{x}{21}$ | (xiv) $\frac{x^2-4}{2x}$ |
| (iii) $\frac{x}{12}$ | (vii) $-\frac{x^2}{8}$ | (xi) $\frac{3}{2x^2}$ | |
| (iv) $\frac{51x}{10}$ | (viii) 0 | (xii) $\frac{2}{x}$ | (xv) x |