8 Graphs of Quadratic Expressions: The Parabola

In Topic 6 we saw that the graph of a linear function such as $y = 2x + 1$ was a straight line. The graph of a function which is not linear therefore cannot be a straight line.

Here, we look at certain kinds of quadratic (non-linear) functions for which the graph is an important geometrical curve called the **PARABOLA** (a curve studied in depth as early as the 3rd century B.C. by the Greeks such as Apollonius).

Parabolas are of course not the only non-linear curves of importance - others being e.g. $x^2 + y^2 = 1$ (circle), $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (ellipse), $xy = 1$ (hyperbola) - but they are the only ones discussed here.

The shape of the parabola will be obtained experimentally i.e., by plotting enough points to see what the curve looks like. Special features of the parabola we desire to discover are: where it cuts the x and y axes, the equation of its *axis of symmetry*, whether it "points" vertically up or down, and the coordinates of its *vertex* $\equiv$ highest or lowest point).

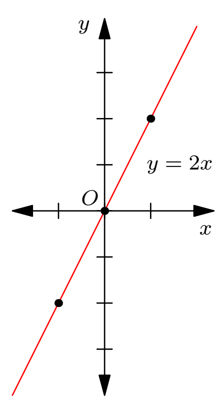
Parabolas occur widely in the world around us, e.g., the path of a projectile (e.g., a drop of water in a fountain, a football which has been kicked) is a parabola.

8.1 Linear Curve (Line)

[Revision]

$x =$	-1	0	1
$y =$	-2	0	2

$\Big\}\ y = 2x$



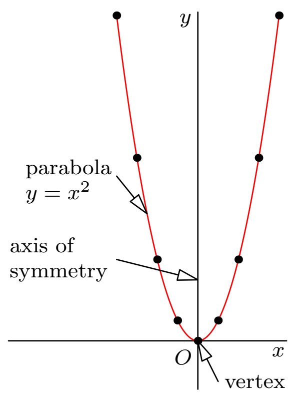
8.2 Non-Linear Curve (e.g., Parabola)

Firstly, let us tabulate some pairs of values of x and y which satisfy $y = x^2$:

$x =$	-4	-3	-2	-1	0	1	2	3	4
$y =$	16	9	4	1	0	1	4	9	16

$\Big\} \; y = x^2$

Now plot the points representing these number-pairs:



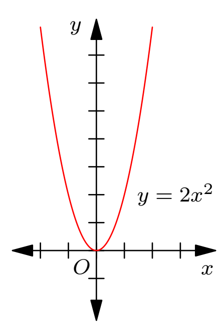
Observe that in the quadratic function $f(x) = x^2$ for the parabola, we have written y instead of $f(x)$ (i.e., $y = f(x)$).

Some similar parabolas which are clearly related to the standard one drawn above are shown below. Notice that $y = -x^2$ points upwards instead of downward and that $y = 2x^2$ has the same general shape and position as $y = x^2$ except that its y-value is always twice as big as the corresponding y-value for $y = x^2$, i.e., it is "thinner" (more elongated).

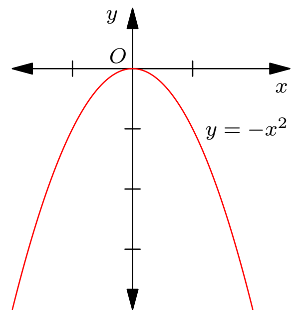
Because x^2 must always be ≥ 0* no matter whether x is positive or negative, the parabola $y = x^2$ can never lie below the x-axis (i.e., the line $y = 0$).

***Note:** ≥ 0 means "greater than or equal to 0" (see 9). Thus, $x^2 \geq 0$ means that x^2 is not negative.

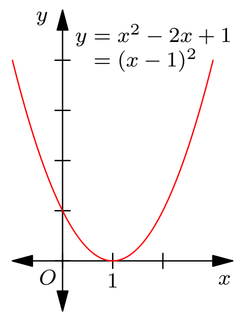
$y = 2x^2$



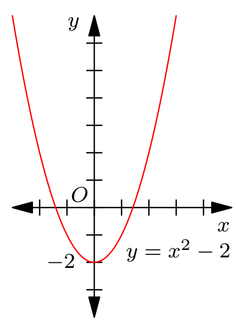
$y = -x^2$



$$\begin{aligned} y &= (x-1)^2 \\ &= x^2 - 2x + 1 \end{aligned}$$



$y = x^2 - 2$



Exercises 8.1:

(i) $y = \frac{1}{2}x^2$

(ii) $y = -3x^2$

(iii) $y = x^2 + 1$

(iv) $y = (x+1)^2$

(v) $y = -(x-2)^2$

(vi) $y = (x-1)^2 - 2$

Sketch all of these.

8.3 More General Parabolas

A graph often gives important information about the function it represents.

Example:

$$\begin{aligned} y &= x^2 - 5x + 6 \\ &= (x-2)(x-3) \end{aligned}$$

This expression happens to have factors, but this won't always be the case. As $y = 0$ when $x = 2$ and $x = 3$, this means that the parabola cuts the x-axis ($y = 0$) at $x = 2$ and $x = 3$. [If the expression in x does not factorize, use the formula in Topic 7, Section 4 to find x when $y = 0$.]

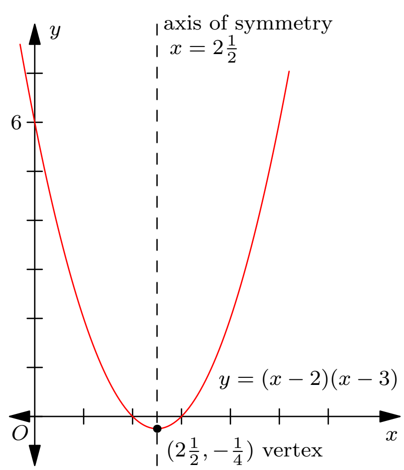
When $x = 0$, the value of y is 6, i.e. the curve cuts the y-axis at $y = 6$.

To get an accurate sketch of the curve, you will have to plot several points.

The axis of symmetry will lie between $x = 2$ and $x = 3$, and it is obvious that $x = 2\frac{1}{2}$ is the axis of symmetry. Putting $x = 2\frac{1}{2}$ in the equation of the parabola we obtain

$$\begin{aligned} y &= \left(2\frac{1}{2}\right)^2 - 5 \times \left(2\frac{1}{2}\right) + 6 \\ &= -\frac{1}{4} \end{aligned}$$

$\therefore$ the vertex is the point $\left(2\frac{1}{2}, -\frac{1}{4}\right)$.



In general, the equation of the axis of symmetry for a parabola of the form $y = ax^2 + bx + c$ is

$$x = -\frac{b}{2a}$$

i.e., for $y = x^2 - 7x + 6$, the equation of the axis of symmetry is

$$\begin{aligned} x &= \frac{-(-7)}{2 \times 1} \\ &= 3\frac{1}{2} \end{aligned}$$

Observe that the equation of the parabola drawn may be written

$$y = \left(x - 2\frac{1}{2}\right)^2 - \frac{1}{4} \qquad\qquad (= x^2 - 5x + 6)$$

which gives us the vertex and axis of symmetry very quickly.

Large values of x in $y = x^2 - 5x + 6$

When x is positive and large, y is also positive and large (symbolically: $x \to \infty$ implies $y \to \infty$).

When x is negative and large, y is **positive** and large (symbolically: $x \to -\infty$ implies $y \to \infty$).

Note: ∞ is the symbol for "infinity". ∞ is **not** a number. $x \to \infty$ means x **approaches** infinity, i.e. x increases beyond the largest positive numbers. Similarly for $x \to -\infty$.

If you have difficulty in (mentally or otherwise) manipulating large numbers, think of $x = 100$ as being "large" enough.

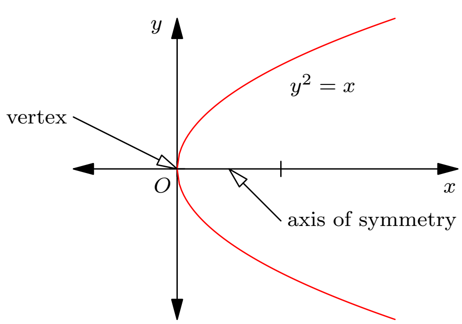
Exercises 8.2:

Sketch

(i) $y = x^2 + x - 2$

(ii) $y = 2x^2 - 4x + 7$

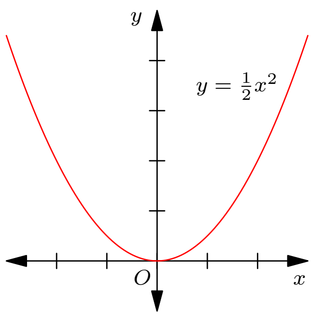
Before leaving this elementary introduction to the parabola with a vertical axis of symmetry, we should notice that there is an analogous treatment for the parabola with a horizontal axis of symmetry. The simplest instance of this kind of parabola is that given by the equation $x = y^2$ for which the graph is



8.4 Answers to Exercises

8.1:

(i)



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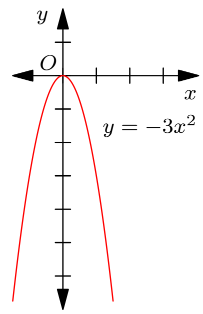

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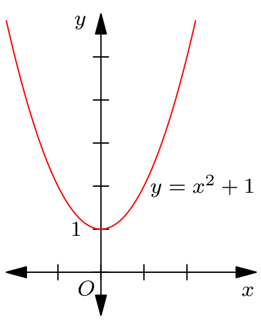

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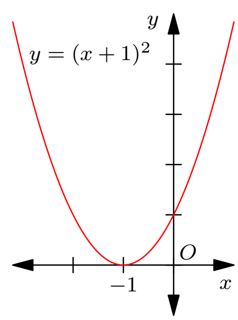

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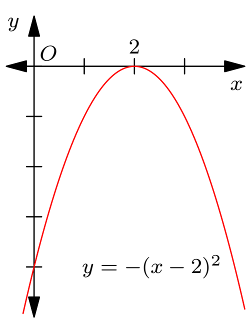

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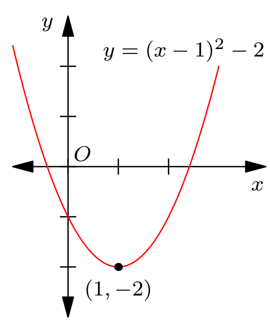
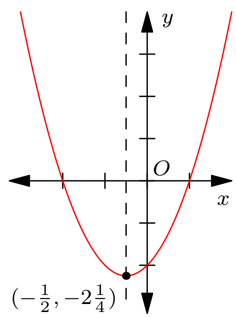


8.2:

(i)



(ii)