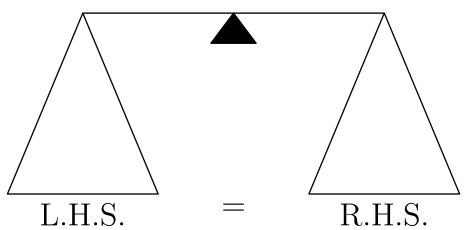
5 Linear Equations

5.1 Solution of a linear Equation with one Variable

A *linear equation* contains no indices greater than 1 of the *variable* x (say), e.g. $3x - 5 = 7$ is a linear equation but $3x^2 - 5 = 7$ is a non-linear equation.

A linear equation can be represented pictorially as a *straight line* – see Topic 6.

The two sides of an equation are like the two sides of a set of scales in equilibrium. Whatever you do to one side you must do to the other side to preserve the equilibrium.



Examples:

(a) Solve $3x - 5 = 7$

Solution:

$$\begin{aligned} 3x - 5 + 5 &= 7 + 5 && \text{adding 5 to each side} \\ 3x &= 12 && \text{simplifying} \\ \frac{3x}{3} &= \frac{12}{3} && \text{dividing both sides by 3} \\ x &= 4 && \text{simplifying} \end{aligned}$$

Check by substituting $x = 4$ in the original equation

i.e.

$$\begin{aligned} \text{L.H.S.} &= 3 \times 4 - 5 = 12 - 5 \\ &= 7 \\ &= \text{R.H.S.} \\ \therefore \quad x &= 4 \quad \text{is the correct solution.} \end{aligned}$$

(b) Solve $4(x-1)+2=2x-5$

$$
\begin{aligned}
\text{Solution} \qquad 4x-4+2 &= 2x-5 \\
4x-2 &= 2x-5 \\
4x-2+2 &= 2x-5+2 \\
4x &= 2x-3 \\
4x-2x &= 2x-3-2x \\
2x &= -3 \\
\frac{2x}{2} &= -\frac{3}{2} \\
x &= -1\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\textbf{Check:} \qquad \text{L.H.S.} &= 4\left(-1\frac{1}{2}-1\right)+2 \qquad \text{when } x=-1\frac{1}{2} \\
&= 4\left(-2\frac{1}{2}\right)+2 \\
&= -10+2 \\
&= -8 \\
\text{R.H.S.} &= 2\left(-1\frac{1}{2}\right)-5 \qquad \text{when } x=-1\frac{1}{2} \\
&= -3-5 \\
&= -8 \\
&= \text{L.H.S.} \\
\therefore \quad x &= -1\frac{1}{2} \quad \text{is the correct solution.}
\end{aligned}
$$

Note: While checking is seldom required in a problem, it is a useful exercise especially if you are asked to use the answer in some further problem. So, make a habit of checking.

Exercises 5.1:

Solve

(i) $3x - 1 = 5$

(ii) $4x + 2 = 9$

(iii) $6 - 2x = 1$

(iv) $5x + 1 = -9$

(v) $3x + 1 = x - 5$

(vi) $6x - 1 = 3 - 2x$

(vii) $2 - 3(x - 2) = x + 4$

(viii) $5(x - 1) + 1 = 2(x - 2)$.

In general, a linear equation in one variable has just **one** *solution, i.e., a unique solution.*

[Examples of linear equations in one variable which do not have a unique solution are

$$3\left(x + \frac{4}{3}\right) = 3x + 4 \quad \text{and} \quad 3x + 4 = 3(x + 4).$$

In the first equation, both sides of the equation are identical and so any value of x will satisfy the equation and hence there will be an "infinite number" of solutions. In the second equation, $3x + 4 = 3x + 12$, which is a contradiction, so no solution is possible.]

A linear equation in more than one variable will **not** *have a unique solution, e.g.* $3x + 4y = 2$ *(see Section 2 below).*

5.2 Linear Equations in two Variables

Examples:

(a) $3x + 4y = 0$ i.e. $y = -\frac{3}{4}x$ has an "infinite number" of solutions which may be tabulated thus:

x	$\cdots$	-1	0	1	$\frac{4}{3}$	2	$\cdots$
y	$\cdots$	$\frac{3}{4}$	0	-$\frac{3}{4}$	-1	-1$\frac{1}{2}$	$\cdots$

For every value of x, there is a value of y, e.g., $x = \frac{4}{3}$, $y = -1$.

(b) $3x + 4y = 2$ i.e. $y = \frac{1}{2} - \frac{3x}{4}$ has a infinite number of solutions, e.g.

x	$\cdots$	-1	0	1	$\frac{4}{3}$	2	$\cdots$
y	$\cdots$	$1\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	$\cdots$

Exercise 5.2: Solve $2x - y = 3$

5.3 Pair of Linear Equations in two Variables

Suppose we are given the equations $3x + 4y = 2$ and $x + y = 3$. Each equation separately has an infinite number of solutions. Taken together, however, these two equations have a unique solution.

The equations are then said to be solved *simultaneously* i.e., "at the same time".

There are two equivalent methods of solution to this problem.

Example:

$$\text{Solve} \quad \left.\begin{array}{ll} 3x + 4y = 2 & \dots\dots\dots\dots\dots\dots (i) \\ x + y = 3 & \dots\dots\dots\dots\dots\dots (ii) \end{array}\right\} \begin{array}{l}\text{simultaneous} \\ \text{linear equations}\end{array}$$

First solution:

$$\begin{array}{rlll} (ii) \times (4): & 4x + 4y & = 12 & \dots\dots\dots\dots\dots\dots (iii) \\ & 3x + 4y & = \ \ 2 & \dots\dots\dots\dots\dots\dots (i) \\ \hline (iii) - (i): & x & = 10 & \end{array}$$

Substitute in (ii):

$$\begin{aligned} 10 + y &= 3 \\ 10 + y - 10 &= 3 - 10 \\ y &= -7 \end{aligned}$$

$\therefore$ the *unique* solution is $\boxed{\begin{aligned} x &= 10 \\ y &= -7 \end{aligned}}$

Always substitute these values of x and y in (i), (ii) to **check** the correctness of the solution.

Second solution: Write (i), (ii) as

$$\begin{aligned} y &= -\frac{3}{4}x + \frac{1}{2} \ldots\ldots\ldots\ldots\ldots (i)' \\ y &= -x + 3 \ldots\ldots\ldots\ldots\ldots\ldots (ii)' \end{aligned}$$

$(i)'$, $(ii)'$ give $-\frac{3}{4}x + \frac{1}{2} = -x + 3\ (= y)$ which we solve as in Topic 5, Section 1.

$$\begin{aligned} -\frac{3}{4}x + \frac{1}{2} - \frac{1}{2} &= -x + 3 - \frac{1}{2} \\ -\frac{3}{4}x &= -x + 2\frac{1}{2} \\ -\frac{3}{4}x + x &= -x + 2\frac{1}{2} + x \\ \frac{1}{4}x &= 2\frac{1}{2} \\ \frac{1}{4}x \times 4 &= 2\frac{1}{2} \times 4 \\ x &= 10 \end{aligned}$$

Substitute for x in $(i)'$: $y = -\frac{3}{4} \times 10 + \frac{1}{2} = -7\frac{1}{2} + \frac{1}{2} = -7$

Substitute for x in $(ii)'$: $y = -10 + 3 = -7$

$\therefore$ the unique solution is $\boxed{\begin{aligned} x &= 10 \\ y &= -7 \end{aligned}}$

[The checking of the solution is inherent in the process.]

Exercises 5.3:

(i) Solve

(a) $2x + 3y = 10,\ 3x - 2y = -11$

(b) $5x - 2y = 1,\ 2x = 4 - y$

(ii) Why do the pairs of equations

(a) $x - y = 5,\ 3x - 3y = 4$

(b) $x - y = 5,\ 3x - 3y = 15$

not have a unique solution?

5.4 Geometrical Interpretation

The *unique* solution of 2 simultaneous **linear** equations gives the *unique* point of intersection of 2 straight **lines**.

See the next Topic, Graphs of Straight Lines.

5.5 Answers to Exercises

5.1:

(i) 2 (iii) $2\frac{1}{2}$ (v) -3 (vii) 1

(ii) $1\frac{3}{4}$ (iv) -2 (vi) $\frac{1}{2}$ (viii) 0

5.2:

$$\left\{\begin{array}{lrrrrrl} x = \ldots, & -1, & 0, & 1, & 1\frac{1}{2}, & 2, & \ldots \\ y = \ldots, & -5, & -3, & -1, & 0, & 1, & \ldots \end{array}\right. \quad \text{are some solutions.}$$

5.3:

(i) (a) $x = -1,\ y = 4$

(b) $x = 1,\ y = 2$

(ii) (a) the two equations are **inconsistent**. (if $x - y = 5$ is true, then $3x - 3y = 3(x - y) = 3 \times 5 = 15$, not 4)

(b) the second equation is exactly 3 times the first equation, i.e., there is only one (independent) equation, i.e., the equations are not **independent.**