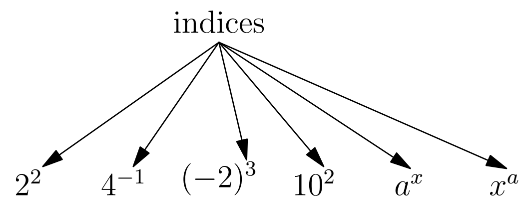
4 Indices

4.1 Introduction



Previously (see Topic 1, Section 6) we have dealt only with indices which were positive or negative integers, or zeros.

Now we turn to indices which are fractions.

4.2 Fractional Indices

The *square root*, written $\sqrt{a}$, of a positive number a, is the number which, when squared (multiplied by itself once) gives a.

$$\text{e.g. } \sqrt{1} = 1, \quad \sqrt{4} = 2, \quad \sqrt{9} = 3$$

Alternatively, the square root may be written with a fractional index as:

$$\boxed{\sqrt{a} = a^{\frac{1}{2}}}$$

Thus,

$$(\sqrt{a})^2 = (a^{\frac{1}{2}})^2 = a$$

i.e.

$$\sqrt{a} \times \sqrt{a} = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$$

For example,

$$\begin{aligned} 9^{\frac{1}{2}} &= \sqrt{9} \qquad (\textit{square root} \text{ of } 9) \\ &= 3 \qquad \text{because } \underbrace{3 \times 3}_{\text{2 factors}} = 9 \end{aligned}$$

Similarly,

$$\begin{aligned} 8^{\frac{1}{3}} &= \sqrt[3]{8} \qquad (\textit{cube root} \text{ of } 8) \\ &= 2 \qquad \text{because } \underbrace{2 \times 2 \times 2}_{\text{3 factors}} = 8 \end{aligned}$$

and

$$\begin{aligned} 8^{-\frac{1}{3}} &= \frac{1}{8^{\frac{1}{3}}} \\ &= \frac{1}{2} \end{aligned}$$

In general, the "k-th root" of a is

$$a^{\frac{l}{k}} = \sqrt[k]{a} \qquad \text{i.e., } \underbrace{a^{\frac{l}{k}} \times a^{\frac{l}{k}} \times \ldots \times a^{\frac{l}{k}}}_{k \text{ factors}} = a$$

[When $k = 2$, we usually write $\sqrt[2]{a}$ merely as $\sqrt{a}$]

> *Exercises 4.1:*
>
> (i) $4^{\frac{1}{2}}$
>
> (ii) $9^{-\frac{1}{2}}$
>
> (iii) $16^{-\frac{1}{4}}$
>
> (iv) $25^{\frac{1}{2}}$
>
> (v) $(32)^{\frac{1}{5}}$

4.3 Multiplication

Recall $a^0 = 1 \quad (a \neq 0) \quad$ (Topic 1, Section 6)

Example:

$$\underbrace{2^3 \times 2^4}_{\text{same base}} = \underbrace{2 \times 2 \times 2}_{3 \text{ times}} \times \underbrace{2 \times 2 \times 2 \times 2}_{4 \text{ times}} = \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{3+4=7 \text{ times}} = 2^7 \quad \overbrace{(= 2^{3+4})}^{\boxed{\text{add indices}}}$$

$$\boxed{a^m \times a^n = a^{m+n}}$$

Exercises 4.2:

(i) $3^2 \times 3^4$

(ii) $5^2 \times 5^{-3}$

(iii) $4^3 \times 4^{-3}$

(iv) $2^2 \times 3^2$

(v) $x^2 \times x^3$

(vi) $x^2y \times x^5y^2$

(vii) $2xy \times 3x^2y^4$

(viii) $3x^2(2x - x^3)$

4.4 Division

Example:

$$\frac{3^3}{3^5} = \frac{\overbrace{3 \times 3 \times 3}^{3 \text{ times}}}{\underbrace{3 \times 3 \times 3 \times 3 \times 3}_{5 \text{ times}}} = \frac{1}{\underbrace{3 \times 3}_{5-3=2 \text{ times}}} = \frac{1}{3^2} = 3^{-2} \qquad \overbrace{(= 3^{3-5})}^{\boxed{\text{subtract indices}}}$$

$$\boxed{\frac{a^m}{a^n} = a^{m-n}}$$

Note:

$$\frac{a}{a} = a^{1-1} = a^0 = 1 \qquad (a \neq 0)$$

Exercises 4.3:

(i) $\frac{2^7}{2^3}$

(ii) $\frac{5^4}{5^4}$

(iii) $\frac{3^{-4}}{3^{-5}}$

(iv) $\frac{(-3)^4}{(-3)^5}$

(v) $\frac{x^7y^5}{x^4y^3}$

(vi) $\frac{15x^5y^2}{3x^5y}$

(vii) $\frac{10xz}{5z}$

(viii) $\frac{a^6}{b^3} \times \frac{b^2}{a^2}$

4.5 $(a^m)^n$

Example:

$$(5^2)^3 = \underbrace{\underbrace{5\times 5}_{2\text{ times}} \times \underbrace{5\times 5}_{2\text{ times}} \times \underbrace{5\times 5}_{2\text{ times}}}_{3\text{ times}} = \underbrace{5\times5\times5\times5\times5\times5}_{2\times3=6\text{ times}} = 5^6 \quad \overbrace{(=5^{2\times 3})}^{\boxed{\text{multiply indices}}}$$

$$\boxed{(a^m)^n = a^{mn}}$$

Let us consider $a^{\frac{p}{k}} = (a^{\frac{l}{k}})^p$ where p, k are integers ($\neq 0$). i.e. the index $\frac{p}{k}$ is a fraction. In particular, we have:

$$\begin{aligned} a^{\frac{2}{3}} &= a^{\frac{1}{3}.2} = (a^{\frac{1}{3}})^2 = \underbrace{a^{\frac{1}{3}} \times a^{\frac{1}{3}}}_{2\text{ factors}} \quad [\text{with } \underbrace{a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}}}_{3\text{ factors}} = a] \\ &= (\sqrt[3]{a})^2 \end{aligned}$$

For example,

$$\text{(i)} \quad 8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 \quad \textbf{or} \quad 8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}$$
$$= 2^2 \qquad\qquad = 2^{3\times\frac{2}{3}} = 2^2$$
$$= 4 \qquad\qquad = 4$$

$$\text{(ii)} \quad 16^{-\frac{3}{2}} = \frac{1}{16^{3/2}} = \frac{1}{(\sqrt{16})^3} \quad \textbf{or} \quad 16^{-\frac{3}{2}} = (2^4)^{-\frac{3}{2}}$$
$$= \frac{1}{4^3} \qquad\qquad = 2^{4\times(-\frac{3}{2})=2^{-6}}$$
$$= \frac{1}{64} \qquad\qquad = \frac{1}{64}$$

Exercises 4.4:

(i) $(2^5)^4$

(ii) $(2^4)^{-\frac{1}{2}}$

(iii) $[(-10)^{\frac{2}{3}}]^3$

(iv) $[(-27)^{\frac{4}{3}}]^{\frac{3}{2}}$

(v) $(4^{\frac{4}{3}})^{\frac{9}{8}}$

(vi) $2^{\frac{2}{3}} \times 4^{\frac{2}{3}}$

(vii) $(x^3y^2)^3$

(viii) $(x^2)^{-\frac{1}{2}}$

(ix) $x^3 \times y^6$

(x) $8^{-\frac{4}{3}}$

Topic Revision Exercises 4.5:

(i) $(3^2)^{-2}$

(ii) $2^{\frac{1}{2}} \times 2^{-\frac{1}{3}}$

(iii) $(2^{\frac{1}{2}})^{-\frac{1}{3}}$

(iv) $\frac{2^{\frac{5}{7}}}{2^{\frac{3}{4}}}$

(v) [Hard] $\left(\frac{81}{16}\right)^{-\frac{3}{4}} \left[= \frac{(3^4)^{-\frac{3}{4}}}{(2^4)^{-\frac{3}{4}}}\right]$

Note: When using the *solidus* or slash (/) for division, **be very careful.**

$$\text{e.g.}\quad \begin{aligned} x/2x &= x \div (2x) \\ &= \tfrac{1}{2} \end{aligned} \qquad \text{not} \quad \begin{aligned} \left(\tfrac{x}{2}\right) x &= \tfrac{x}{2} \times x \\ &= \tfrac{x^2}{2} \end{aligned}$$

Refer to similar comment in Topic 3, section 2.

Warning! Care should be taken when calculating roots of negative numbers. e.g.

$$\begin{aligned} (-8)^{\frac{1}{3}} &= \sqrt[3]{-8} \\ &= -2 \qquad \text{since } \underbrace{(-2) \times (-2) \times (-2)}_{3 \text{ times}} = -8 \end{aligned}$$

but $(-4)^{\frac{1}{2}} = \sqrt{-4}$ has no value in the system of real numbers (see Topic 9). [$\sqrt{-4}$ is an example of a *complex number*. The system of complex numbers is an extension of the system of real numbers.]

4.6 Answers to Exercises

4.1:

(i) 2

(ii) $\frac{1}{3}$

(iii) $\frac{1}{2}$

(iv) 5

(v) 2

4.2:

(i) 3^6

(ii) $5^{-1} = \frac{1}{5}$

(iii) $1 (= 4^0)$

(iv) $(2 \times 3)^3 = 6^2 = 36 = 4 \times 9$

(v) x^5

(vi) $x^7 y^3$

(vii) $6x^3 y^5$

(viii) $6x^3 - 3x^5$

4.3:

(i) 2^4

(ii) 1

(iii) 3

(iv) $(-3)^{-1} = -\frac{1}{3}$

(v) $x^3 y^2$

(vi) $5y$

(vii) $2x$

(viii) $\frac{a^4}{b}$

4.4:

(i) 2^{20}

(ii) $2^{-2} - \frac{1}{4}$

(iii) 100

(iv) 729

(v) $4^{\frac{3}{2}} = (2^2)^{\frac{3}{2}} = 2^3 = 8$

(vi) 4 (Note $4^{\frac{2}{3}} = (2^2)^{\frac{2}{3}} = 2^{\frac{4}{3}}$)

(vii) $x^9 y^6$

(viii) $x^{-1} = \frac{1}{x}$

(ix) $(xy^2)^3$

(x) $\frac{1}{16}$

4.5:

(i) $\frac{1}{81}$

(ii) $2^{\frac{1}{6}}$

(iii) $2^{-\frac{1}{6}}$

(iv) $2^{-\frac{1}{28}}$

(v) $\frac{8}{27}$